Chapter One

CUBES AND CUBE ROOTS

1.1: Cubes

2^3 means $2 \times 2 \times 2$

$3^3 = 3 \times 3 \times 3$

$4^3 = 4 \times 4 \times 4$

The cube of the number is a number multiplied by itself three times.

In general, $a^3 = a \times a \times a$.

Example 1

What is the value of 5^3?

Solution

$$5^3 = 5 \times 5 \times 5$$
$$= 25 \times 5$$
$$= 125$$

Example 2

Find the cube of 1.4.

Solution

$$(1.4)^3 = 1.4 \times 1.4 \times 1.4$$
$$= 1.96 \times 1.4$$
$$= 2.744$$

Exercise 1.1

1. Find the cube of each of the following:
   (a) 6 (b) 1.6 (c) 3.2
2. What is the value of $(2.17)^3$?
3. Find the value of each of the following:
   (a) $(0.4)^3$ (b) $(0.5)^3$ (c) $(1.1)^3$ (d) $(0.03)^3$
4. Find the volume of a cube of side 3 cm.
5. Find the value of each of the following:
   (a) $(ab)^3$ (b) $(xy)^3$ (c) $(2b)^3$

1.2: Use of Tables to find Cubes

Cubes of numbers can be read directly from tables. The technique of reading cubes using tables is similar to that used for tables of squares and square roots.

Table 1.1 is an extract of tables of cubes.

Table 1.1

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.000	1.030	1.061	1.093	1.125	1.158	1.191	1.225	1.260	1.295	3	7	10	13	16	20	23	26	30
1.1	1.331	1.368	1.405	1.443	1.482	1.521	1.561	1.602	1.643	1.685	4	8	12	16	20	24	28	31	35
1.2	1.728	1.772	1.816	1.861	1.907	1.953	2.000	2.048	2.097	2.147	5	9	14	19	23	28	33	37	42
1.3	2.197	2.248	2.300	2.353	2.406	2.460	2.515	2.571	2.628	2.686	5	11	16	22	27	33	38	43	49
1.4	2.744	2.803	2.863	2.924	2.986	3.049	3.112	3.177	3.242	3.308	6	13	19	25	31	38	44	50	56
1.5	3.375	3.443	3.512	3.582	3.652	3.724	3.796	3.870	3.944	4.020	7	14	21	29	36	43	50	57	64
1.6	4.096	4.173	4.252	4.331	4.411	4.492	4.574	4.657	4.742	4.827	8	16	24	32	41	49	57	65	73
1.7	4.913	5.000	5.088	5.178	5.268	5.359	5.452	5.545	5.640	5.735	9	18	27	37	46	55	64	73	82
1.8	5.832	5.930	6.029	6.128	6.230	6.332	6.435	6.539	6.645	6.751	10	20	31	41	51	61	71	82	92
1.9	6.859	6.968	7.078	7.189	7.301	7.415	7.530	7.645	7.762	7.881	11	23	34	45	57	68	79	91	102
2.0	8.000	8.121	8.242	8.365	8.490	8.615	8.742	8.870	8.999	9.129	13	25	38	50	63	75	88	100	113
2.1	9.261	9.394	9.528	9.664	9.800	9.938	10.078	10.218	10.360	10.503	14	28	41	55	69	83	97	110	124
2.2	10.648	10.794	10.941	11.090	11.239	11.391	11.543	11.697	11.852	12.009	15	30	45	60	76	91	106	121	136
2.3	12.167	12.326	12.487	12.649	12.813	12.978	13.144	13.312	13.481	13.652	16	33	49	66	82	99	115	132	148
2.4	13.824	13.998	14.172	14.349	14.527	14.706	14.887	15.069	15.253	15.438	18	36	54	72	90	108	126	143	161
2.5	15.625	15.813	16.003	16.194	16.387	16.581	16.777	16.975	17.174	17.374	19	39	58	78	97	117	136	155	175
2.6	17.576	17.780	17.985	18.191	18.400	18.610	18.821	19.034	19.249	19.465	21	42	63	84	105	126	147	168	189
2.7	19.683	19.903	20.124	20.346	20.571	20.797	21.025	21.254	21.485	21.718	23	45	68	90	113	136	158	181	203
2.8	21.952	22.188	22.426	22.665	22.906	23.149	23.394	23.640	23.888	24.138	24	49	73	97	121	146	170	194	219
2.9	24.389	24.642	24.897	25.154	25.412	25.672	25.934	26.198	26.464	26.731	26	52	78	104	130	156	182	208	234
3.0	27.000	27.271	27.544	27.818	28.094	28.373	28.653	28.934	29.218	29.504	28	56	83	111	139	167	195	223	250
3.1	29.791	30.080	30.371	30.664	30.959	31.256	31.554	31.855	32.157	32.462	30	59	89	119	148	178	208	237	267
3.2	32.768	33.076	33.386	33.698	34.012	34.328	34.646	34.966	35.288	35.611	32	63	95	126	158	190	221	253	284
3.3	35.937	36.265	36.594	36.926	37.260	37.595	37.933	38.273	38.614	38.958	34	67	101	134	163	201	235	269	302
3.4	39.304	39.652	40.002	40.354	40.708	41.064	41.422	41.782	42.144	42.509	36	71	107	142	178	214	249	285	320
3.5	42.87	43.24	43.61	43.99	44.36	44.74	45.12	45.50	45.88	46.27	4	8	11	15	19	23	26	30	34
3.6	46.66	47.05	47.44	47.83	48.23	48.63	49.03	49.43	49.84	50.24	4	8	12	16	20	24	28	32	36
3.7	50.65	51.06	51.48	51.90	52.31	52.73	53.16	53.58	54.01	54.44	4	8	13	17	21	25	29	34	38
3.8	54.87	55.31	55.74	56.18	56.62	57.07	57.51	57.96	58.41	58.86	4	9	13	18	22	27	31	35	40
3.9	59.32	59.78	60.24	60.70	61.16	61.63	62.10	62.57	63.04	63.52	5	9	14	19	23	28	33	37	42
4.0	64.00	64.48	64.96	65.45	65.94	66.43	66.92	67.42	67.92	68.42	5	10	15	20	25	29	34	39	44
4.1	68.92	69.43	69.93	70.44	70.96	71.47	71.99	72.51	73.03	73.56	5	10	15	21	26	31	36	41	46
4.2	74.09	74.62	75.15	75.69	76.23	76.77	77.31	77.85	78.40	78.95	5	11	16	22	27	32	38	43	49
4.3	79.51	80.06	80.62	81.18	81.75	82.31	82.88	83.45	84.03	84.60	6	11	17	23	28	34	40	45	51
4.4	85.18	85.77	86.35	86.94	87.53	88.12	88.72	89.31	89.92	90.52	6	12	18	24	30	36	41	47	53
4.5	91.12	91.73	92.35	92.96	93.58	94.20	94.82	95.44	96.07	96.70	6	12	19	25	31	37	43	50	56
4.6	97.34	97.97	98.61	99.25	99.90	100.54	101.19	101.85	102.50	103.16	6	13	19	26	32	39	45	52	58
4.7	103.82	104.49	105.15	105.82	106.50	107.17	107.85	108.53	109.22	109.90	7	14	20	27	34	41	47	54	61
4.8	110.59	111.28	111.98	112.68	113.38	114.08	114.79	115.50	116.21	116.93	7	14	21	28	35	42	49	56	63
4.9	117.65	118.37	119.10	119.82	120.55	121.29	122.02	122.76	123.51	124.25	7	15	22	29	37	44	51	59	66
5.0	125.000	125.75	126.51	127.26	128.02	128.79	129.55	130.32	131.10	131.87	8	15	23	31	38	46	53	61	69
5.1	132.65	133.43	134.22	135.01	135.80	136.59	137.39	138.19	138.99	139.80	8	16	24	32	40	48	56	64	71
5.2	140.61	141.42	142.24	143.06	143.88	144.70	145.53	146.36	147.20	148.04	8	17	25	33	41	50	58	66	74
5.3	148.88	149.72	150.57	151.42	152.27	153.13	153.99	154.85	155.72	156.59	9	17	26	34	43	51	60	69	77
5.4	157.46	158.34	159.22	160.10	160.99	161.88	162.77	163.67	164.57	165.47	9	18	27	36	44	53	62	71	80

Example 3

Use tables to find the cube of each of the following:

(a) 1.8 (b) 2.12 (c) 3.254 (d) 0.76

Solution

(a) $(1.8)^3 = 5.832$

(b) $(2.12)^3 = 9.528$

(c) $(3.254)^3 = 34.454$

$= 34.45$ (4 s.f.)

(d) $(0.76)^3 = (7.6 \times 10^{-1})^3$

$$= \left(\frac{7.6}{10}\right)^3$$
$$= \frac{438.9}{1\,000}$$
$$= 0.4389$$

Exercise 1.2

1. Use mathematical tables to find the cube of each of the following:
   (a) 8.3 (b) 1.01 (c) 2.504 (d) 0.87 (e) 15.45
2. Use tables to find:
   (a) $(4.06)^3$ (b) $(6.312)^3$ (c) $(0.0912)^3$
   (d) $(381.7)^3$ (e) $(2.1534)^3$ (f) $(5.3679)^3$
3. A cubic building block measures 21 cm. Find its volume.
4. A cubic water tank has sides of length 2.143 m. What is the capacity of the tank in litres?

1.3: Cube Roots using Factor Method

$$2 \times 2 \times 2 = 2^3$$
$$= 8$$

The cube root of 8 is 2, usually written as;

$$\sqrt[3]{8} = 2$$

Similarly, $27 = 3 \times 3 \times 3$

$$\therefore \sqrt[3]{27} = 3$$

In general, the cube root of a number is the number that is multiplied by itself three times to get the given number.

Example 4

Evaluate: $\sqrt[3]{216}$

Solution

$$\sqrt[3]{216} = \sqrt[3]{2 \times 2 \times 2 \times 3 \times 3 \times 3}$$
$$= 2 \times 3$$
$$= 6$$

Example 5

Find: $\sqrt[3]{a^6 b^3}$

Solution

$$\sqrt[3]{a^6 b^3} = \sqrt[3]{a^6} \times \sqrt[3]{b^3}$$

$= \sqrt[3]{a \times a \times a \times a \times a \times a} \times \sqrt[3]{b \times b \times b}$
$= a \times a \times b$
$= a^2b$

Example 6

The volume of a cube is 1 000 cm^3. What is the length of the cube?

Solution

Volume of the cube, $V = l^3$

$l^3 = 1000$

$l = \sqrt[3]{1\,000}$
$= \sqrt[3]{10 \times 10 \times 10}$
$= 10$

$\therefore$ The length of the cube is 10 cm.

Exercise 1.3

1. (a) $\sqrt[3]{64}$ (b) $\sqrt[3]{125}$ (c) $\sqrt[3]{3\,375}$
   (d) $\sqrt[3]{\frac{27}{64}}$ (e) $\sqrt[3]{0.512}$ (f) $\sqrt[3]{2\frac{93}{125}}$
2. Evaluate:
   (a) $\sqrt[3]{a^3b^3}$ (b) $\sqrt[3]{x^6y^6}$ (c) $\sqrt[3]{w^6y^3}$ (d) $\sqrt[3]{27x^3y^9}$
3. Find:
   (a) $\sqrt[3]{\frac{0.064 \times 125}{343}}$ (b) $\sqrt[3]{\frac{135.01 \times 21.952}{6.859}}$
   (c) $\sqrt[3]{(x - y)(x + y)}$, when x = 76 and y = 49
4. The volume of a sphere is given by $\frac{4}{3}\pi r^3$. Find the radius of a sphere whose volume is 1047.816 cm^3. (Take π to be $\frac{22}{7}$)
5. The volume of material used to make a cube is 1 728 cm^3. What is the length of the side of the cube?
6. The volume of water in a measuring cylinder reads 200 cm^3. When a cube is immersed into the water, the cylinder reads 543 cm^3. Find:
   (a) the volume of the cube.
   (b) the length of the side of the cube.
7. A metallic cuboid measuring 16 cm by 8 cm by 4 cm was melted. The material was then used to make a cube. What was the length of the cube?

Chapter Two

RECIPROCALS

2.1: Reciprocals of Numbers by Division

If the reciprocal of a number a is b, this implies that $a \times b = 1$. This means that $b = \frac{1}{a}$.

For example, the reciprocal of 2 is $\frac{1}{2}$, and $2 \times \frac{1}{2} = 1$.

The reciprocal of 4 is $\frac{1}{4}$ and that of 20 is $\frac{1}{20}$.

The bigger a number, the smaller its reciprocal. The reciprocal of decimal numbers is got through division.

Example 1

Find the reciprocal of 3.6.

$$\frac{1}{3.6} = \frac{1}{3.6} \times \frac{10}{10} = \frac{10}{36} = \frac{5}{18}$$

$$\begin{array}{r} 0.277\ldots \\ 18\overline{)50} \\ -\underline{36} \\ 140 \\ -\underline{126} \\ 140 \\ -\underline{126} \\ 14 \end{array}$$

$\therefore$ The reciprocal of 3.6 is $0.2\dot{7}$

Exercise 2.1

1. Find the reciprocal of each of the following integers:
   (a) 3 (b) 8 (c) 12 (d) 28 (e) 256
2. Find the reciprocal of each of the following decimal numbers using division:
   (a) 2.5 (b) 3.8 (c) 4.2 (d) 8.6 (e) 9.4

2.2: Reciprocals of Numbers from Tables

Reciprocals can be found using reciprocal tables.

Example 2

Find the reciprocal of 3.665 using reciprocal tables.

Solution

Using reciprocal tables, the reciprocal of 3.665 is $0.2732 - 0.0004 = 0.2728$

Example 3

Find the reciprocal of 45.8.

Solution

45.8 is first written is standard form as 4.58×10^1.

$$\text{then, } \frac{1}{45.8} = \frac{1}{4.58 \times 10^1}$$
$$= \frac{1}{10} \times \frac{1}{4.58}$$
$$= \frac{1}{10} \times 0.2183$$
$$= 0.02183$$

The reciprocal of 45.8 from the tables is 0.02183.

Example 4

Find the reciprocal of 0.0236.

Solution

0.0236 in standard form is 2.36×10^{-2}

$$\frac{1}{0.0236} = \frac{1}{2.36 \times 10^{-2}}$$
$$= \frac{1}{10^{-2}} \times \frac{1}{2.36}$$
$$= 10^2 \times 0.4237$$
$$= 42.37$$

The reciprocal of 0.0236 from the tables is 42.37

Exercise 2.2

1. Find the reciprocal of each of the following numbers using reciprocal tables:

   (a) 56.2 (b) 102.4 (c) 0.234
   (d) 0.0458 (e) 4368 (f) 0.002978

2. Use reciprocal tables to work out each of the following:

   (a) $\frac{1}{0.0125} + \frac{1}{12.5}$ (b) $\frac{3}{0.364}$ (c) $\frac{17}{0.051} + \frac{3}{0.0027}$ (d) $\frac{4}{0.375} - \frac{5}{37.5}$

3. Use mathematical tables to evaluate each of the following:

   (a) $\frac{1}{\sqrt{0.2468}}$ (b) $\frac{6}{(2.437)^2}$ (c) $\frac{1000}{92.56}$ (d) $100 \times \frac{1}{0.5789}$

   (e) $\frac{7}{\sqrt[3]{6.859}}$ (f) $\left(\frac{4}{0.2976}\right)^3$ (g) $\frac{1}{29.43} + \sqrt{2.358}$ (h) $3.045^2 + \frac{1}{\sqrt{49.24}}$

   (i) If $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, find f, given that u = 0.5 and v = 0.8.

Chapter Three

INDICES AND LOGARITHMS

3.1: Indices

The power to which a number is raised is called the index (*plural* indices)
2^5 is the short form for $2 \times 2 \times 2 \times 2 \times 2$.
5 is called the power or index and 2 the base.
In general, if n is a positive number, a^n means $a \times a \times a$ .... n times, where the number a occurs n times.

For example;

$$a^3 = a \times a \times a$$

$$2^3 = 2 \times 2 \times 2 = 8$$

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$

$$4^4 = 4 \times 4 \times 4 \times 4 = 256$$

3.2: Laws of Indices

Consider the following:

(i) $a^5 \times a^2 = (a \times a \times a \times a \times a) \times (a \times a)$
$= a^7$
$= a^{(5+2)}$ [Sum of the two powers]

(ii) $a^5 \text{ x } a^2 = \dfrac{(a \times a \times a \times \cancel{a} \times \cancel{a})}{\cancel{a} \times \cancel{a}}$
$= a^3$
$= a^{(5-2)}$ [Difference of the two powers]

(iii) $(a^5)^2 = a^5 \text{ x } a^5$
$= a^{10}$
$= a^{(5 \text{ x } 2)}$ [Product of the two powers]

The three examples illustrate the following laws of indices;

(i) $a^m \text{ x } a^n = a^{(m+n)}$
(ii) $a^m \div a^n = a^{(m-n)}$
(iii) $(a^m)^n = a^{mn}$

Note:

For these laws to hold, the base must be the same (common base).

The Zero Index

Consider the expression $a^n \div a^n$.

$$a^n \div a^n = \frac{\cancel{a^n}^{1}}{\cancel{a^n}_{1}}$$
$$= 1$$

For example, $a^3 \div a^3 = \frac{\cancel{a^3}^{1}}{\cancel{a^3}_{1}}$

$$= 1$$

Also, using the second law of indices;

$$a^3 \div a^3 = a^{(3-3)}$$
$$= a^0$$

This implies that $a^0 = 1$.

Example 1

Evaluate: (a) $2^6 \times 2^4$ (b) $3^{10} \div 3^4$ (c) $(5^2)^3$

Solution

(a) $2^6 \times 2^4 = 2^{(6+4)}$
$= 2^{10}$
$= 1\,024$

(b) $3^{10} \div 3^4 = 3^{(10-4)}$
$= 3^6$
$= 729$

(c) $(5^2)^3 = 5^2 \times 5^2 \times 5^2$
$= 5^{(2\times 3)}$
$= 5^6$
$= 15\,625$

Example 2

Solve: (a) $(2^x)^3 = 2^{12}$ (b) $(4^3)^x = 64$

Solution

(a) $2^{(x)3} = 2^{12}$
$\therefore 3x = 12$
$x = 4$

(b) $(4^3)^x = 64$
$4^{3x} = 4^3$
$\therefore 3x = 3$
$x = 1$

Negative Indices

Consider the expression $a^{-n} \times a^n$.

Using first law of indices, $a^{-n} \times a^n = a^0$.

Since $a^{-n} \times a^n = a^0$;

$$a^{-n} = \frac{a^0}{a^n}$$

But $a^0 = 1$

Therefore, $a^{-n} = \frac{1}{a^n}$

$$\text{Consider } 2^3 \div 2^4 = \frac{2^3}{2^4}$$

$$= \frac{\cancel{2}^1 \times \cancel{2}^1 \times \cancel{2}^1}{2 \times \cancel{2}_1 \times \cancel{2}_1 \times \cancel{2}_1}$$

$$= \frac{1}{2^1}$$

Using second law of indices;

$$2^3 \div 2^4 = 2^{(3-4)}$$
$$= 2^{-1}$$

Therefore, $2^{-1} = \frac{1}{2^1}$

$$\text{Similarly, } a^3 \div a^5 = a^{(3-5)}$$
$$= a^{-2}$$
$$= \frac{1}{a^2}$$

Example 3

Evaluate: (a) $2^3 \times 2^{-3}$ (b) $\left(\frac{2}{3}\right)^{-2}$

Solution

(a) $2^3 \times 2^{-3} = 2^{(3+-3)}$
$= 2^{(3-3)}$
$= 2^0$
$= 1$

(b) $\left(\frac{2}{3}\right)^{-2} = \left(\frac{1}{\frac{2}{3}}\right)^2$
$= \left(\frac{1}{\frac{4}{9}}\right)$
$= \left(\frac{9}{4}\right)$
$= 2\frac{1}{4}$

Alternatively,

$$\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$$

Fractional Indices

Fractional indices are written in fractional form, i.e., $\frac{p}{q}$, where p and q are integers and q is not equal to zero. The integers should not have a common factor other than one, i.e., $\frac{p}{q}$ should be in its simplest form.

If $a^2 = b$, then a is called the square root of b, written as; $\sqrt{b}$.

If $a^3 = b$, then a is called the third root (or cube root) of b, written as $\sqrt[3]{b}$.

If $a^4 = b$, then a is called the fourth root of b, written as $\sqrt[4]{b}$.

In general, if $a^n = b$, a is called the n^{th} root of b, written as $\sqrt[n]{b}$.

For example; $\sqrt{16} = 4$

$\sqrt[3]{8} = 2$

$\sqrt[4]{16} = 2$

Consider the number $4^{\frac{1}{2}}$

$$4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\left(\frac{1}{2}+\frac{1}{2}\right)} = 4^1 = 4$$

Therefore, $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 2 \times 2$

$= 4$

Hence, $4^{\frac{1}{2}} = 2$

$= \sqrt{4}$

Similarly, $4^{\frac{1}{3}} \times 4^{\frac{1}{3}} \times 4^{\frac{1}{3}} = 4^{\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)}$

$= 4^{\left(\frac{3}{3}\right)}$

$= 4^1$

$= 4$

Therefore, $4^{\frac{1}{3}} = \sqrt[3]{4}$

Also, $4^{\frac{1}{4}} \times 4^{\frac{1}{4}} \times 4^{\frac{1}{4}} \times 4^{\frac{1}{4}} = 4^{\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)}$

$= 4^{\left(\frac{4}{4}\right)}$

$= 4^1$

$= 4$

Therefore, $4^{\frac{1}{4}} = \sqrt[4]{4}$

In general, $a^{\frac{1}{n}} = \sqrt[n]{4}$

Note:

$$\left(a^{\frac{1}{n}}\right)^m = a^{\frac{1}{n}\text{x}m} = a^{\frac{m}{n}}$$

Therefore, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

Example 4

Evaluate: (a) $27^{\frac{1}{3}}$ (b) $16^{\frac{3}{4}}$ (c) $4^{-\frac{1}{2}}$

Solution

(a) $27^{\frac{1}{3}} = \sqrt[3]{27}$
$= 3$

(b) $16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3$
$= 2^3$
$= 8$

Alternatively, $16^{\frac{3}{4}} = \left(16^3\right)^{\frac{1}{4}}$
$= \left(4\,096\right)^{\frac{1}{4}}$
$= \sqrt[4]{4\,096}$
$= 8$

(c) $4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}}$
$= \frac{1}{\sqrt{4}}$
$= \frac{1}{2}$

Note:

In (b), it is easier to take the root first, then the cube.

Exercise 3.1

1. Find the numerical value of:

(a) $9^{\frac{1}{2}}$ (b) $25^{-\frac{1}{2}}$ (c) $36^{-\frac{1}{2}}$ (d) $35\,769^{0}$ (e) $81^{0.5}$

(f) $36^{1.5}$ (g) $8^{-\frac{5}{3}}$ (h) $27^{\frac{1}{3}}$ (i) $16^{2.5}$ (j) $100^{4.5}$

(k) $64^{\frac{4}{3}}$ (l) $256^{0.75}$ (m) $729^{\frac{1}{3}}$ (n) $\left(49^{2}\right)^{-\frac{1}{2}}$ (p) $\left(10^{3}\right)^{\frac{5}{3}}$

(q) $(3^{-2})^{3}$ (r) $\left(216^{2}\right)^{\frac{1}{3}}$ (s) $243^{\frac{2}{3}}$ (t) $4\,096^{-\frac{3}{4}}$ (u) $64^{-\frac{1}{6}}$

2. Simplify each of the following expressions. You may leave your answer in index form:

(a) $3^{3} \times 3^{4}$ (b) $5^{2} \times 5^{-3}$

(c) $2^{3} \times 2^{5} \times 2^{-4}$ (d) $3^{1.5} \times 3^{-0.75} \times 3^{0.25}$

(e) $6^{2} \times 7^{-4} \times (8^{-2})^{2} \times 6^{3} \times 7^{2} \times 8^{4}$ (f) $20^{-3} \times 25^{2} \times 20^{3} \times 25^{-4}$

(g) $7^{-3} \times 8^{4} \times 7^{2} \times 8^{-3}$ (h) $27^{\frac{1}{3}} \times 9^{\frac{1}{2}}$

(i) $512^{\frac{2}{3}} \times 8^{\frac{1}{3}} \div 2^{6}$ (j) $\left(\frac{3}{4}\right)^{2} \times \left(\frac{3}{4}\right)^{3}$

(k) $\left(\frac{6}{7}\right)^{-2} \times \left(\frac{7}{6}\right)^{-2}$ (l) $\dfrac{2^{-2} \times (2^{2})^{-6}}{2^{-4} \times 2^{-6}}$

(m) $\dfrac{20^{-2} \times 20^{-2} \times 20^{-2}}{20}$ (n) $2^{-6} \times 5^{\frac{1}{2}} \times 2^{7} \div 2^{-4}$

3. Simplify:

(a) $x^{2} \times x^{4}$ (b) $y^{3} \times y^{6}$ (c) $n^{13} \times n^{-7}$

(d) $a^{9} \times a^{3}$ (e) $a^{-2} \times a^{4}$ (f) $(2n)^{3} \times (2n)^{5}$

(g) $x^{16} \div n^{12}$ (h) $a^{15} \div a^{14}$ (i) $a^{2} \times b^{2} \times a^{4}$

(j) $b^{2} \times x^{2} \times b^{11}$ (k) $b^{3} \times n^{2} \times b^{0} \times n^{3}$

(l) $a^{5} \times c^{2} \times a^{3} \times b^{3} \times a^{4}$ (m) $a^{-2} \times b^{2} \times a^{-8} \times b^{-4}$

(n) $r^{3} \times r^{2}t^{2} \times r^{-3}t^{-2}$ (p) $x^{2}y^{2} \times x^{2}y^{5} \times x^{-3}y^{-9}$

(q) $(xyt)^{n} \div (xyt)^{2}$ (r) $\dfrac{a^{n} \times a^{m}}{a^{-n} \times a^{-m}}$ (s) $\dfrac{a^{x} \times a^{2x}}{a^{x} \times a^{x} \times a^{x}}$

(t) $\dfrac{x^{2}y^{3} \times x^{4}y^{2}}{x^{3}y^{4}}$ (u) $\dfrac{a^{4}b^{3} \times a^{2}c^{4} \times b^{-2}c^{3}}{a^{-3}b^{3}c^{3}}$

4. Simplify:

(a) $\dfrac{25y^9k^{-7}}{81y^7k^9}$ (b) $\dfrac{12a^5b^3}{4a^3b}$ (c) $\dfrac{3^{-4}a^4b^6}{3a^5b^8}$

(d) $\dfrac{a^{-a}b^2 \times b^{-a} \times c^{-a}}{a^{-2a} \times b^{-2a} \times c^{-2a}}$ (e) $\dfrac{(a^{-n})^m \times a^{-2m}}{a^{mn} \times a^{2m}}$ (f) $\dfrac{rs^2t^3 \times r^4s^5t^{-6} \times r^7s^8t^{-9}}{r^{10}s^{11}t^{-12} \times r^{13}s^{-14}t^{15}}$

(g) $\dfrac{\left(9a^2\right)^{\frac{1}{3}}}{6^4} \times \dfrac{\left(4b^2\right)^{\frac{1}{3}}}{5a}$ (h) $\dfrac{\left(16a^2\right)^{\frac{1}{2}} \times \left(36a^4\right)^{-\frac{1}{2}}}{2a^{\frac{1}{2}} \times 5a^{\frac{3}{2}} \times 8a^{\frac{9}{4}}}$

(i) $\sqrt[3]{\dfrac{27x^3y^9}{x^6y^3}}$ (j) $\dfrac{\sqrt[5]{\left(32y^{10}k^{15}2^{10}\right)}}{\sqrt[4]{\left(16y^{12}k^{16}2^4\right)}}$

5. Solve each of the following equations:

(a) $(3^{2x})^3 = 3^4 \text{ x } 3^8$ (b) $(7^5)^x = (7^4)^x \div 7^2$ (c) $(3^{2x})^4 = 81$

(d) $4^{5x} \div (2^{3x})^2 = 256$ (e) $9^{2x} = 729$ (f) $2^{8x} = 512$

(g) $(7^4)^{2x} = (7^4)^3$ (h) $(5^3)^a \times (5^8)^a = 5^{72}$ (i) $9^{4x} \div 3^{2x} = 2\,187$

(j) $(5^{2c})^3 = (5^4)^c \times 625$ (k) $2^{\frac{x}{4}} = 8$ (l) $3^{(2x-5)} = 27$

(m) $3^{4x} \div 3^{-7} = 3^{15}$ (n) $2 \times 3^x = 162$ (p) $\dfrac{64^{2y} \div 16^y}{128^y \times 4^{2y}}$

6. Write in the form a^n:

(a) (i) 8 (ii) 4 (iii) 2 (iv) 1

(v) $\frac{1}{2}$ (vi) $\frac{1}{4}$ (viii) $\frac{1}{4}$ (where a is 2)

(b) (i) 27 (ii) 9 (iii) 3 (iv) 1

(v) $\frac{1}{3}$ (vi) $\frac{1}{9}$ (vii) $\frac{1}{27}$ (where a is 3)

(c) (i) 1 000 (ii) 100 (iii) 10 (iv) 1

(v) $\frac{1}{10}$ (vi) $\frac{1}{100}$ (vii) $\frac{1}{1\,000}$ (where a is 10)

Logarithms

Consider some powers of 3:

$3^0 = 1$

$3^1 = 3$

$3^2 = 9$

$3^3 = 27$

$3^4 = 81$

The indices 0, 1, 2, 3, 4 ... are called the **logarithms** of the corresponding numbers to **base** 3.

For example; logarithm of 9 to **base** 3 is 2.

logarithm of 81 to **base** 3 is 4.

These are usually written in short form as;

$\log_3 9 = 2$

$\log_3 81 = 4$

Copy and complete the table below:

Index form	*Logarithm form*
$2^2 = 4$	$\text{Log}_2 4 = 2$
$2^3 = 8$	
$2^4 = 16$	
	$\text{Log}_2 128 = 7$
$4^2 = 16$	
$5^2 = 25$	
	$\text{Log}_5 625 = 4$
$10^1 = 10$	
$10^2 = 100$	

Generally, the expression $a^m = n$ is written as $\log_a n = m$. $a^m = n$ is the **index notation** while $\log_a n = m$ is the **logarithmic notation.**

Example 5

Write in logarithm form:

(a) $2^4 = 16$ (b) $9^{\frac{1}{2}} = 3$ (c) $b^n = m$

Solution

(a) If $2^4 = 16$, then $\log_2 16 = 4$ (b) If $9^{\frac{1}{2}} = 3$, then $\log_9 3 = \frac{1}{2}$

(c) If $b^n = m$, then $\log_b m = n$

Example 6

Write in index form:

(a) $\log_{10} 1\,000 = 3$ (b) $\log_3 81 = 4$ (c) $\log_b m = n$

Solution

(a) If $\log_{10} 1000 = 3$, then $10^3 = 1\ 000$.

(b) If $\log_3 81 = 4$, then $3^4 = 81$.

(c) If $\log_b m = n$, then $b^n = m$.

Exercise 3.2

1. Write in logarithm form:

(a) $3^2 = 9$ (b) $2^4 = 16$ (c) $3^3 = 27$

(d) $2^5 = 32$ (e) $3^4 = 81$ (f) $5^3 = 125$

(g) $10^0 = 1$ (h) $2^{10} = 1\ 024$ (i) $a^n = b$

2. Write each of the following in index form:

(a) $\log_2 8 = 3$ (b) $\log_4 16 = 2$ (c) $\log_5 125 = 3$

(d) $\log_{10} 8 = x$ (e) $\log_b a = c$ (f) $\log_3 27 = 3$

(g) $\log_6 216 = 3$ (h) $\log_x 40 = y$ (i) $\log_4 6 = y$

(j) $\log_y x = 2$ (k) $\log_{10} 10\ 000 = 4$ (l) $\log_2 16 = 4$

Standard Form

Consider the following:

$12 = 1.2 \times 10^1$

$120 = 1.2 \times 10^2$

$1\ 200 = 1.2 \times 10^3$

$0.12 = 1.2 \times \frac{1}{10}$

$= 1.2 \times 10^{-1}$

$0.286 = 2.86 \times \frac{1}{10}$

$= 2.86 \times 10^{-1}$

$0.0074 = 7.4 \times \frac{1}{1\ 000}$

$= 7.4 \times 10^{-3}$

Any number can be written in the form $A \times 10^n$, where A is a number between 1 and 10 (10 not included) or $1 \leq A < 10$, and n is an integer. When written in this way, a number is said to be in **standard form.**

Write each of the following numbers in standard form:

(i) 26 (ii) 357 (iii) 4 068 (iv) 15 000 000

(v) 0.031 (vi) 0.00215 (vii) 0.005012 (viii) 0.000000152

(ix) 100 000 (x) $\frac{46}{100\ 000}$

3.3: Powers of 10 and Common Logarithms

1, 10, 100 and 1000 can be expressed as powers of 10 as follows;

1 $= 10^0$
10 $= 10^1$
100 $= 10^2$
1000 $= 10^3$

The indices 0, 1, 2 and 3 are called the logarithms to base 10 of 1, 10,100 and 1 000 respectively. Since the base is 10, they are referred to as common logarithms. What are the common logarithms of 10 000, 100 000 and 1 000 000?

In this book, we refer to logarithms to base 10 as logarithms. In short, we write the logarithm of a number, say 100, to base 10 as $\log_{10} 100 = 2$. This is simply written as $\log 100 = 2$.

Most numbers cannot be expressed as exact powers of 10, for example 70. Note that 70 lies between 10 or 10^1 and 100 or 10^2. Therefore, its logarithm to base 10 is a number between 1 and 2. Calculations to 4 decimal places have shown that;

70 $= 10^{1.8451}$ 55 $= 10^{1.7404}$
121 $= 10^{2.0828}$ 347 $= 10^{2.5403}$
962 $= 10^{2.9832}$

Logarithms of numbers to base 10 are listed in tables of logarithms. Below is an extract of such.

Logarithms to Base 10

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1.1	.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	27	30	34
1.2	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
1.3	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
1.4	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1.5	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1.6	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
1.7	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1.8	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1.9	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
2.0	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2.1	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2.2	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
2.3	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
2.4	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
2.5	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
2.6	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
2.7	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
2.8	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
2.9	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
3.0	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13

The technique of reading logarithm tables is similar to that applied for tables of squares, square roots and reciprocals.

Example 7

Read from tables the logarithm of:

(a) 2.56 (b) 3.752 (c) 28.5

Solution

(a) Log 1 = 0 and log 10 = 1. We therefore expect log 2.56 to be between 0 and 1. From tables, log 2.56 = 0.4082. This means $10^{0.4082} = 2.56$.

(b) Log 3.75 = 0.5740
Difference of 2 is 0.0002
$\therefore$ log 3.752 = 0.5740 + 0.0002
= 0.5742
Therefore, $10^{0.5742} = 3.752$

(c) $28.5 = 2.85 \times 10^1$ in standard form.
log 2.85 = 0.4548, i.e., $10^{0.4548} = 2.85$
$\log 10^1 = 1$
$28.5 = 2.85 \times 10^1 = 10^{0.4548} \times 10^1 = 10^{1.4548}$
Therefore, log 28.5 = 1.4548.

We can read the logarithms of numbers between 1 and 10 directly from tables. For a number greater than 10, we proceed as follows:

(i) Express the number in standard form, $A \times 10^n$. Then n will be the whole number part of the logarithm.
(ii) Read the logarithm of A from the tables, which gives the decimal part of the logarithm.

Example 8

Find the logarithm of:

(a) 379 (b) 5 280 000

Solution

(a) $379 = 3.79 \times 10^2$
log 3.79 = 0.5786
Therefore, log 379 = 2.5786

(b) $5\,280\,000 = 5.28 \times 10^6$
log 5.28 = 0.7226
Therefore, log 5 280 000 = 6.7226

The whole number part of the logarithm is called **the characteristic** and the decimal part the **mantissa**.

State the characteristics and the mantisa in the examples above.

Exercise 3.3

1. Use logarithm tables to express each of the following numbers in the form 10^x:

(a) 8.2	(b) 1.47	(c) 4.73	(d) 7.25
(e) 9.83	(f) 5.672	(g) 8.137	(h) 3.142
(i) 2.718	(j) 3.333	(k) 12.3	(l) 59.7
(m) 82.9	(n) 72	(p) 96.1	(q) 431.5
(r) 7 924	(s) 1 025	(t) 1 913	(u) 4 937
(v) 273.7	(w) 475 000	(x) 3 910 000	(y) 958 312

3.4: Logarithms of Positive Numbers Less Than 1

Consider the logarithms to base 10 of 0.034.

Logarithm of 0.034 cannot be read directly from the tables of logarithm. We therefore proceed as follows;

(i) Express 0.034 in standard form, i.e., $A \times 10^n$.

(ii) Read the logarithm of A and add to n.

Thus, $0.034 = 3.4 \times 10^{-2}$

From the tables, $\log 3.4 = 0.5315$. This means $3.4 = 10^{0.5315}$

Therefore, $3.4 \times 10^{-2} = 10^{0.5315} \times 10^{-2}$

$= 10^{0.5315 + -2}$

$= 10^{-2} + 0.5315$

Log of $0.034 = -2 + 0.5315$

$-2 + 0.5315$ is written as $\bar{2}.5315$ and read as **bar two point five, three, one, five.** The negative sign is written above 2 to emphasise that **only the characteristic is negative.**

Example 9

Find the logarithm of:

(a) 0.57 (b) 0.00063

Solution

(a) $0.57 = 5.7 \times 10^{-1}$

$= 10^{0.7559} \times 10^{-1}$

$= 10^{(-1 + 0.7559)}$

$= 10^{\bar{1}.07559}$

Therefore, $\log 0.57 = \bar{1}.7559$

(b) $0.0063 = 6.3 \times 10^{-3}$

$= 10^{0.7993} \times 10^{-3}$

$= 10^{-3} + 0.7993$

$= 10^{\bar{3}.7993}$

Therefore, log 0.0063 is $\bar{3}.7993$

Antilogarithms

If the logarithm of a number is known, we can use tables of antilogarithms to find the number. Finding antilogarithms is the reverse process of finding the logarithm of a number.

Consider the case of finding the number whose logarithm is 0.6107, i.e. $10^{0.6107}$. This number lies between 1 and 10.

In the table of antilogarithms, the antilogarithm of 0.610 = 4.074

The difference for 7 is 0.007

Therefore, the antilogarithm of 0.6107 $= 4.074 + 0.007$
$= 4.081$

Find the numbers whose logarithms are 0.5490, 2.6107 and 3.5026.

It is also possible to use logarithm tables to find antilogarithms or the numbers whose logarithms are known (some mathematical tables do not include tables of antilogarithms). Using logarithm tables, the antilogarithms of 0.5490, 2.6107 and 3.5026 can be found as follows:

(i) Look for 0.5490 in the main columns of the tables. This appears in the row beginning with 35 and under the column headed by 4.

(ii) We know $10^{0.5490}$ is between 1 and 10. Therefore, $10^{0.5490} = 3.540$.

Similarly, $10^{2.6107} = 10^2 \times 10^{0.6107}$
$= 10^2 \times 4.08$
$= 408$

Note:

The answer is different (in the fourth significant figure) from the one obtained using antilogarithm tables. Since all four-figure tables give only approximate values, this sort of discrepancy is not unusual.

Consider $10^{3.5026} = 10^3 \times 10^{0.5026}$. In this case, the number 0.5026 does not appear in the main columns. The number nearest to (and slightly less than) it is 0.5024, which corresponds to 3.18. The difference is 2, which does not appear in the difference columns.

In this case, we pick either 1 or 3 as they are equally close to 2.

1 gives $10^{0.5026} = 3.181$, and,

3 gives $10^{0.5026} = 3.182$.

If the difference you are looking for does not appear in the difference columns, then pick the one nearest to it.

Therefore, in this case, $10^{3.5026} = 10^3 \times 10^{0.5026}$
$= 10^3 \times 3.181$
$= 3\ 181$ (or 3 182)

Read from the table on page 16 the antilog of:

(i) 0.1461 (ii) 0.2487 (iii) 1.4900 (iv) 2.4835

Example 10

Find the number whose logarithm is :

(a) $\bar{2}.3031$ (b) $\bar{4}.5441$

Solution

(a) Let the number be x

$$\begin{aligned} x &= 10^{\bar{2}.3031} \\ &= 10^{(-2+0.3031)} \\ &= 10^{-2} \times 10^{0.3031} \\ &= 10^{-2} \times 2.01 \\ &= \frac{1}{100} \times 2.01 \\ &= \frac{2.01}{100} \\ &= 0.0201 \end{aligned}$$

(b) Let the number be x

$$\begin{aligned} x &= 10^{\bar{4}.5441} \\ &= 10^{(-4+0.5441)} \\ &= 10^{-4} \times 10^{0.5441} \\ &= 10^{-4} \times 3.500 \\ &= \frac{1}{10\,000} \times 3.500 \\ &= \frac{3.500}{10\,000} \\ &= 0.00035 \end{aligned}$$

3.5: Applications of Logarithms

Multiplication and Division from the Laws of Indices

$10^m \times 10^n = (10^{(m+n)}$

and $10^m \div 10^n = 10^{(m-n)}$

The above relationships enable us to use logarithms to perform multiplication and division. For example, to evaluate 357×47.9, we express 357 and 47.9 as powers of 10, as below;

$357 = 10^{2.5527}$, $47.9 = 10^{1.6803}$

$$\begin{aligned} \text{Therefore, } 357 \times 47.9 &= 10^{2.5527} \times 10^{1.6803} \\ &= 10^{(2.5527+1.6803)} \\ &= 10^{4.2330} \\ &= 10^4 \times 10^{0.2330} \end{aligned}$$

Using logarithms or antilogarithm tables, we get the antilog of 0.2330 and multiply by 10^4, i.e., $10^4 \times 1.71$.

Therefore, $357 \times 47.9 = 17\,100$.

Note:
The actual value of 357×47.9 by long multiplication is 17100.3.
The discrepancy is small and can be neglected.

Similarly, the evaluation of $864 \div 136$ is carried out as follows:

$$\begin{aligned} 864 \div 136 &= 10^{2.9365} \div 10^{2.1335} \\ &= 10^{(2.9365 - 2.1335)} \\ &= 10^{0.8030} \end{aligned}$$

Using antilogarithm tables, $10^{0.8030} = 6.353$
Therefore, $864 \div 136 = 6.353$.
Mathematical problems such as the ones above are usually arranged as below:

Number	Standard form	Logarithm
357	3.57×10^2	2.5527
47.9	4.79×10^1	1.6803 +
17 100	1.71×10^4 ←	4.2330

Number	Standard form	Logarithm
864	8.64×10^2	2.9365
136	1.36×10^2	2.1335 –
6.353	6.353×10^0 ←	0.8030

Note:
The base is 10.
With practice, the standard form column is omitted.

Example 11
Use logarithm tables to evaluate:

(a) $\dfrac{456 \times 398}{271}$ (b) 3.14^2 (c) 8.36^3

Solution

(a) $\dfrac{456 \times 398}{271}$

No.	log
456	2.6590
398	2.5999 +
	5.2589
271	2.4330 –
6.697×10^2	2.8259

Therefore, $\dfrac{456 \times 398}{271} = 6.697 \times 10^2 = 669.7$

(b) $3.14^2 = 3.14 \times 3.14$

No.	log
3.14^2	0.4969×2
9.858×10^0	0.9938

Therefore, $3.14^2 = 9.858 \times 10^0$
$= 9.858$

(c) 8.36^3

No.	log
8.36^3	0.9222×3
5.842×10^2	2.7666

Therefore, $8.36^3 = 5.842 \times 10^2$
$= 584.2$

Example 12

Evaluate $\dfrac{415.2 \times 0.0761}{135}$

Solution

Number	Logarithm
415.2	2.6182
0.0761	$\bar{2}.8814\ +$
	1.4996
135	$2.1303\ -$
2.341×10^{-1}	$\bar{1}.3693$

Therefore, $\dfrac{415.2 \times 0.0761}{135} = 2.341 \times 10^{-1}$
$= 0.2341$

Example 13

Evaluate $\bar{2}.49 + \bar{3}.53$

Solution

We write $\bar{2}.49$ and $\bar{3}.53$ in the long form. i.e.;

$\bar{2}.49 = -2 + 0.49$
$\bar{3}.53 = -3 + 0.53$

$$\text{Then, } \bar{2}.49 + 3.53 = \begin{array}{r} -2 + 0.49 \\ -3 + 0.53\ + \\ \hline -5 + 1.02 \\ \hline \end{array} = {}^{-}4 + 0.02$$

$= \bar{4}.02$

Therefore. $2.49 + \bar{3}.53 = \bar{4}.02$

Exercise 3.4

Use mathematical tables to evaluate:

1. (a) 251×367 (b) $4\,192 \times 3\,078$
   (c) 21.47×362.1 (d) 7.32×199
   (e) $26.1 \times 91.2 \times 45.7$ (f) $33.2 \times 172 \times 44.32$
   (g) $1\,527 \times 3\,196 \times 4\,157$ (h) $7.312 \times 49.45 \times 157.2$
   (i) $16.31 \times 152.1 \times 3\,290$ (j) $100 \times 245 \times 175 \times 396$
2. (a) $1\,500 \div 750$ (b) $2\,145 \div 560$ (c) $412 \div 241$
   (d) $882 \div 144$ (e) $3\,612 \div 452$ (f) $1\,111 \div 222$
   (g) $3.142 \div 2.718$ (h) $43.1 \div 3.17$ (i) $9\,250 \div 4\,312$
   (j) $6\,380 \div 2\,137$
3. (a) $\dfrac{291 \times 681}{372}$ (b) $\dfrac{634 \times 436 \times 688}{784}$ (c) $\dfrac{3\,041 \times 3\,211}{2\,112}$
   (d) $\dfrac{788 \times 576}{675}$ (e) $\dfrac{294 + 578}{368 - 275}$ (f) $\dfrac{1\,024 \times 6\,561}{4\,096 \times 729}$
   (g) $\dfrac{839 + 672}{762 + 393}$ (h) $\dfrac{3\,267 - 37}{2\,236 + 89}$ (i) $\dfrac{29^2 \times 33^3}{64^4}$
   (j) $\dfrac{18^2 \times 391^4}{15^3 \times 56^4}$
4. (a) $\dfrac{876.9}{61.2 \times 3.85}$ (b) $\dfrac{518.3}{29.5 \times 714}$ (c) $\dfrac{48.35 \times 125.3}{39.3 \times 50.4}$
   (d) $\dfrac{3.74 \times 7.82}{5.4}$ (e) $\dfrac{399.6}{15.2 \times 4.83 \times 1.98}$ (f) $\dfrac{734.4}{25.1 \times 6.34 \times 3.94}$
   (g) $\dfrac{94.7 \times 16.45}{12.5 \times 8.93}$ (h) $\dfrac{(4.48)^3}{(3.16)^2}$ (i) $\dfrac{(171.5)^3}{56.3 \times 26.98}$
   (j) $\dfrac{(79.36)^3}{8.2 \times (9.2)^2}$.
5. Find the logarithms, to base 10, of:
   (a) 0.149 (b) 0.2843 (c) 0.3520 (d) 0.4286
   (e) 0.0694 (f) 0.0485 (g) 0.0239 (h) 0.000376
   (i) 0.0000784 (j) 0.0000523
6. Find the number whose logarithm to base 10 is:
   (a) $\bar{1}.1080$ (b) $\bar{1}.2480$ (c) $\bar{2}.3927$ (d) $\bar{2}.5403$
   (e) $\bar{3}.6503$ (f) $\bar{4}.8938$ (g) $\bar{3}.9750$ (h) $\bar{2}.5658$
   (i) $\bar{5}.4533$ (j) $\bar{3}.6821$

7. Calculate:

(a) $\bar{2}.37 + 1.20$ (b) $\bar{5}.63 + \bar{2}.57$ (c) $\bar{2}.34 + \bar{4}.30$

(d) $\bar{3}.79 - \bar{2}.24$ (e) $\bar{7}.12 - \bar{5}.54$ (f) $5.27 - \bar{2}.73$

8. Use logarithms to evaluate:

(a) 0.9063×3.387 (b) 0.5060×0.05707

(c) $36.65 \times 0.4163 \times 0.007022$ (d) $26.68 \div 255.4$

(e) $3.404 \div 628$ (f) $\dfrac{0.0075 \times 0.8181}{000.509}$ (g) $\dfrac{77.9}{0.988 \times 9\,100}$

(h) $\dfrac{2.89 \times 5.27}{62.25 \times 1.908}$ (i) $\dfrac{34.53 \times 361.6}{343.7 \times 615.8}$ (j) $\dfrac{91.3 \div 18.26}{75.4 \div 12.09}$

Roots

We can use logarithm tables to find the roots of numbers. Consider the expressions:

(i) $\sqrt{892}$ (ii) $\sqrt[3]{407.6}$ (iii) $\sqrt{0.945}$ (iv) $\sqrt[3]{0.0618}$

The solutions are found as follows:

(i) $\sqrt{892} = 892^{\frac{1}{2}} = \left(10^{2.9504}\right)^{\frac{1}{2}} = 10^{1.4752}$

Therefore, $\sqrt{892} = 2.986 \times 10^1$
$= 29.86$

The working can be arranged as follows:

No.	log
$\sqrt{892}$	$2.9504 \times \frac{1}{2}$
2.986×10^1	1.4752

Therefore, $\sqrt{892} = 29.86$

(ii) $\sqrt[3]{407.6}$

No.	log
$\sqrt[3]{407.6}$	$2.6103 \times \frac{1}{3}$
7.415×10^0	0.8701

Therefore, $\sqrt[3]{407.6} = 7.415$

(iii) $\sqrt{0.945} = \left(9.45 \times 10^{-1}\right)^{\frac{1}{2}} = (10^{\bar{1}.9754} \times \frac{1}{2})$

In order to divide $\bar{1}.9754$ by 2, we rewrite the logarithm in such a way that the negative characteristic is exactly divisible by 2 (if we were looking for the n^{th} root, we would arrange for the characteristic to be exactly divisible by n).

$$\begin{aligned}\bar{1}.9754 &= -1 + 0.9754 \\ &= -2 + 1.9754\end{aligned}$$

$$\begin{aligned}\text{Therefore, } \tfrac{1}{2}(\bar{1}.9754) &= \tfrac{1}{2}(-2 + 1.9754) \\ &= -1 + 0.9877 \\ &= \bar{1}.9877\end{aligned}$$

$$\begin{aligned}\text{Therefore, } \sqrt{0.945} &= 9.720 \times 10^{-1} \\ &= 0.9720\end{aligned}$$

(iv) $\sqrt[3]{0.0618}$

Number	logarithm
$\sqrt[3]{0.0618}$	$\bar{2}.7910 \times \frac{1}{3}$
	$= (\bar{3} + 1.7910) \times \frac{1}{3}$
3.954×10^{-1}	$= \bar{1}.5970$

Therefore, $\sqrt[3]{0.0618} = 0.3954$

Exercise 3.5

1. Divide each of the following logarithms by: (i) 2 (ii) 3 (iii) 4:
   (a) 0.8938 (b) 1.8624 (c) $\bar{2}.9754$
   (d) $\bar{1}.7076$ (e) $\bar{1}.8538$ (f) 2.3502
   (g) 2.1644 (h) $\bar{3}.4928$ (i) $\bar{3}.6946$
   (j) 4.7938
2. Use logarithm tables to find (i) the square root and (ii) the cube root of:
   (a) 478 (b) 2 461 (c) 3 572
   (d) 4 683 (e) 0.0346 (f) 0.00457
   (g) 0.0067 (h) 0.00072 (i) 78.039
   (j) 361.472

3. Use tables to evaluate:

(a) $\sqrt{\left(\dfrac{3.45 \times 16.7}{31.5}\right)}$

(b) $\sqrt{\left(\dfrac{3.142 \times 2.718}{6.49 \times 81.2}\right)}$

(c) $\dfrac{41.56 \times 52.3}{\sqrt{42.88}}$

(d) $\sqrt[3]{\left(\dfrac{1.794 \times 0.038}{12.43}\right)}$

(e) $\sqrt[3]{\dfrac{39.51 \times 614}{0.758}}$

(f) $\sqrt[3]{\left(\dfrac{4862 \times 725}{6437 \times 1024}\right)}$

(g) $\sqrt[4]{\left(\dfrac{6\,978 \times 25.1}{132.7}\right)}$

Chapter Four

GRADIENT AND EQUATIONS OF STRAIGHT LINES

4.1: Gradient

A technician is to climb to the top of a wall 3 metres high for repair work. This is shown in the figure below:

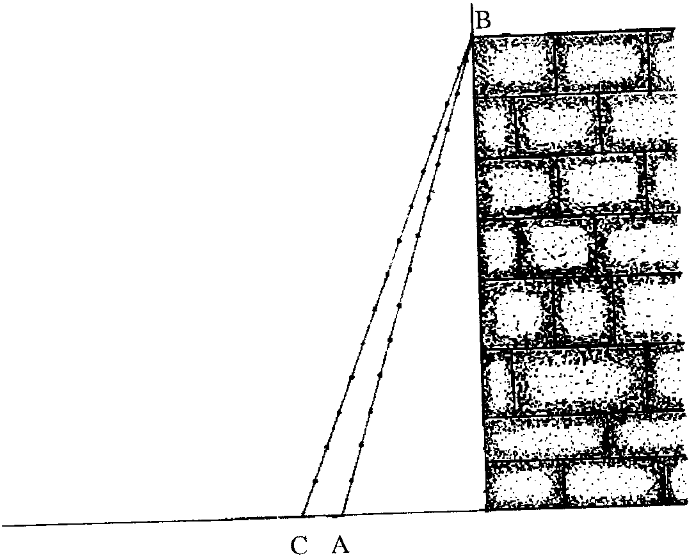


Fig. 4.1

Let AB and CB be two ladders leaning against the wall. If he has the option of using either ladder, which one will be easier?

The distance from the feet of the ladders to the wall are 2 m and 2.5 m respectively. If the technician uses ladder AB, he will have to move through a horizontal distance of 2 m. For every one metre moved horizontally, the corresponding vertical displacement is $1\frac{1}{2}$ or $\frac{3}{2}$ metres.

The ratio $\dfrac{\text{vertical distance}}{\text{horizontal distance}} = \dfrac{3}{2} = 1.5$

For ladder CB, the corresponding ratio $= \dfrac{3}{2.5} = 1.2$

Notice that the ratio is higher for the steeper ladder.

The ratio $\frac{\text{vertical distance}}{\text{horizontal distance}}$, which measures steepness or slope, is known as **gradient.**

Gradients of straight lines are determined in the same way. For example, in figure 4.2, OP is a straight line through the origin. P(4, 3) is a point on the line. The gradient of line OP is;

$$\frac{PQ}{OQ} = \frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{3}{4}$$

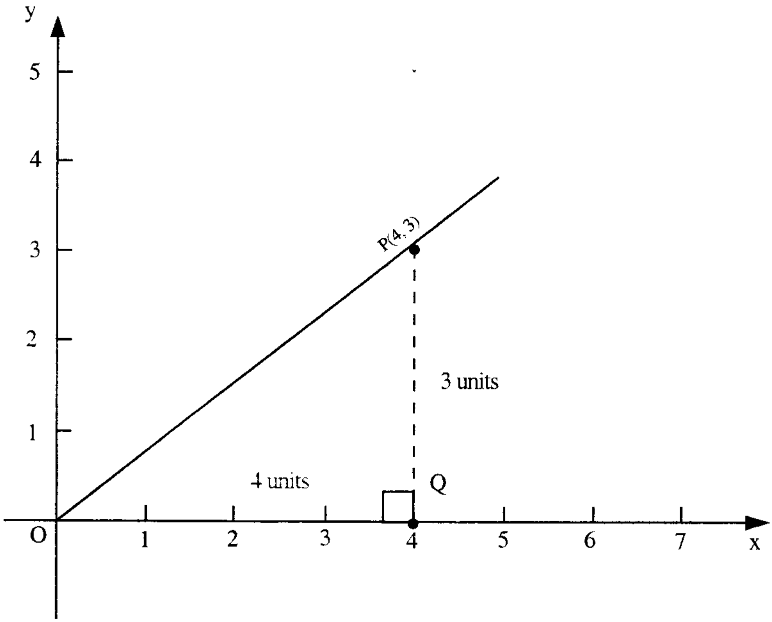


Fig. 4.2

Example 1

Find the gradient of line BP in figure 4.3:

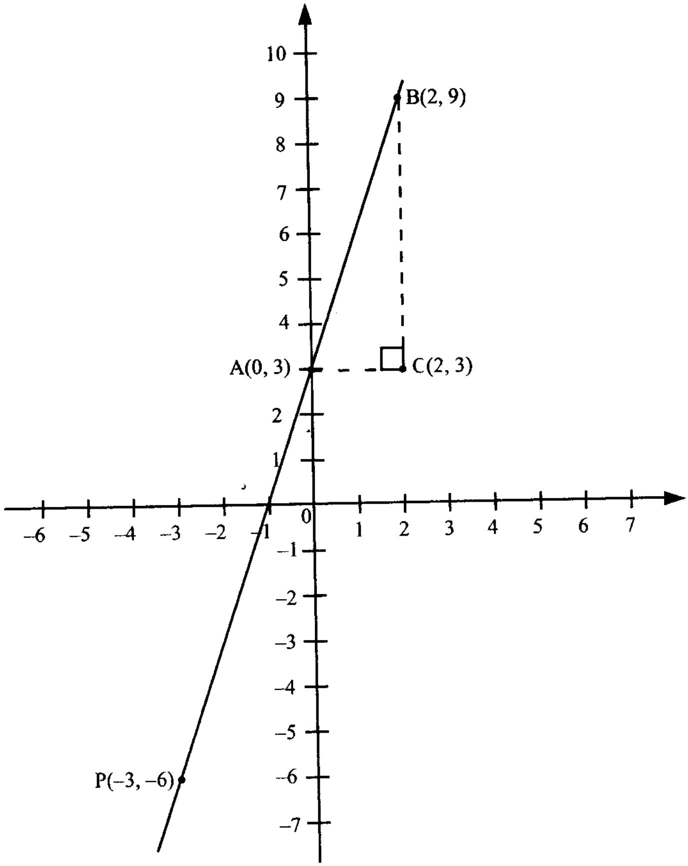


Fig 4.3

Solution

Take any two convenient points on the line, say A(0, 3) and **B(2, 9)**.

$$\text{Gradient of AB} = \frac{BC}{AC}$$

$$= \frac{9-3}{2-0}$$

$$= \frac{6}{2}$$

$$= 3$$

Determine the gradient of the same line using the points P(–3, –6) and B(2, 9).

Figure 4.4 shows a general line through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. The gradient of line PQ is given by;

$$\frac{\text{change in y co-ordinates}}{\text{corresponding change in x co-ordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$$

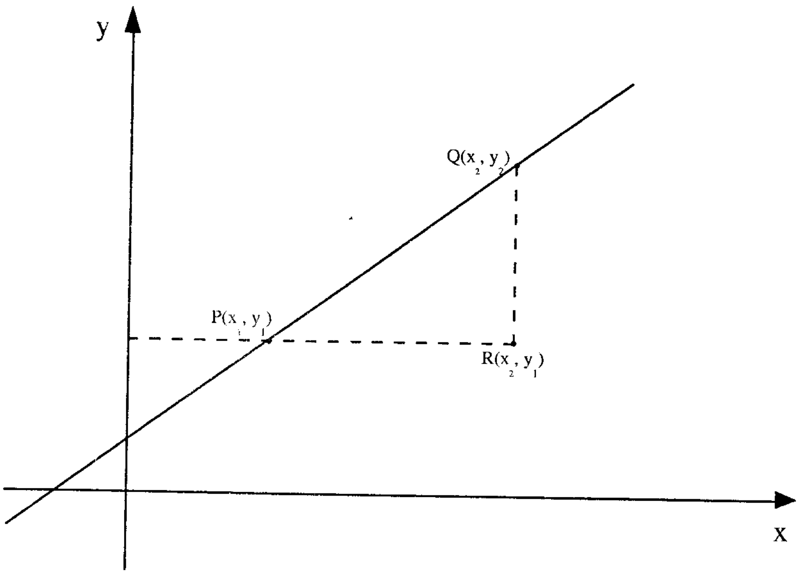


Fig. 4.4

Note:

(i) If an increase in the x co-ordinate also causes a corresponding increase in the y co-ordinate, the gradient is **positive.**

(ii) If an increase in the x co-ordinate causes a decrease in the value of the y co-ordinate, the gradient is **negative.**

(iii) If, for an increase in the x co-ordinate, there is no change in the value of the y co-ordinate, the gradient is **zero.**

(iv) For a vertical line, the gradient is not defined because the horizontal change is zero. Determination of the gradient would entail division by zero, which is not defined.

Example 2

Find the gradient of the line PQ in figure 4.5:

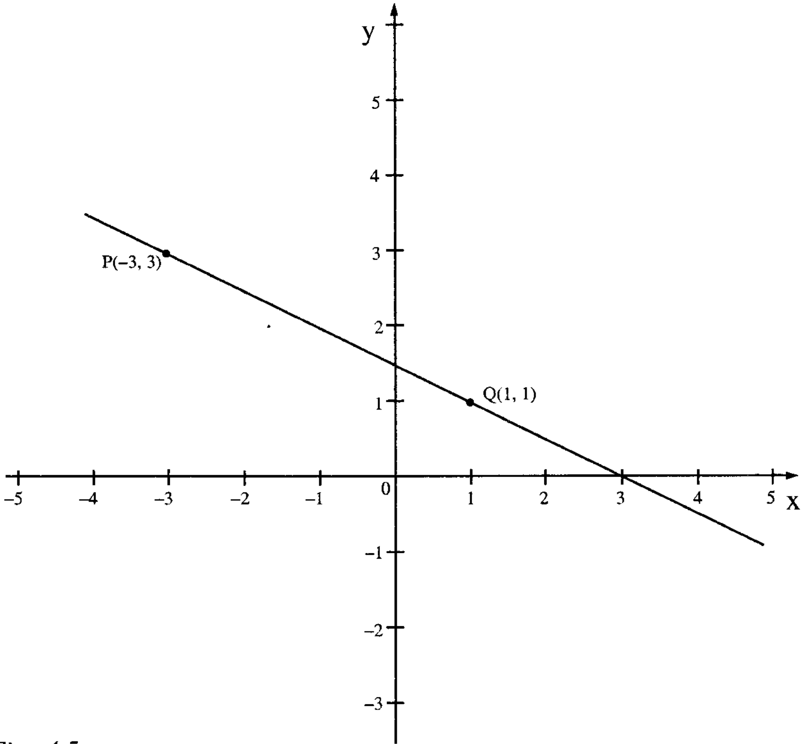


Fig. 4.5

Solution

$$\text{Gradient of PQ} = \frac{\text{change in y co-ordinates}}{\text{corresponding change in x co-ordinates}}$$

$$\text{Change in y co-ordinates} = 1 - 3$$
$$= -2$$

$$\text{Change in x co-ordinates} = 1 - (-3)$$
$$= 4$$

$$\text{Gradient} = -\tfrac{2}{4}$$
$$= -\tfrac{1}{2}$$

Work out the same problem by subtracting the co-ordinates of Q from those of P.

Example 3

Find the gradient of the line y = 3x + 2.

Solution

To find the gradient of a line, we need to find any two points on the line. In y = 3x + 2, choose any two convenient values of x and find the corresponding values of y.

For example, when $x = 0, y = 3 \times 0 + 2 = 2$.

Therefore, the point (0, 2) lies on the line.

When $x = 2, y = 3 \times 2 + 2 = 8$. Point (2, 8) also lies on the line.

Thus, the gradient of the line is $\frac{8-2}{2-0} = \frac{6}{2}$

$= 3$

Exercise 4.1

1. In figure 4.6, name the lines whose gradient is:
   (a) positive (b) negative (c) zero (d) undefined

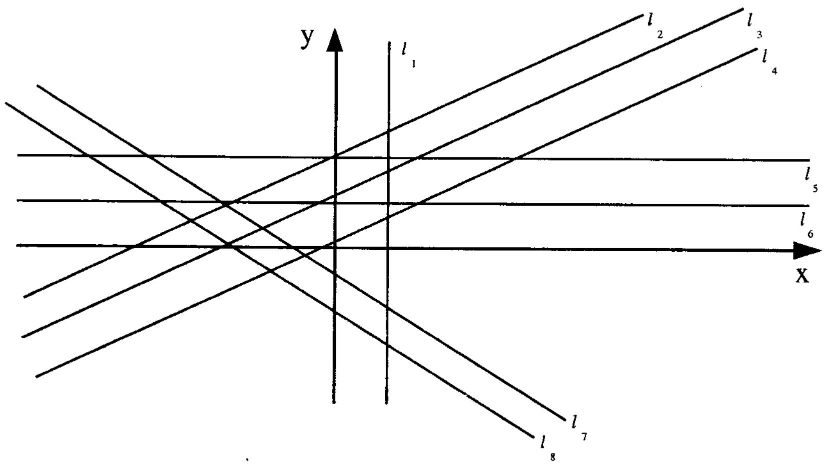


Fig. 4.6

2. For each of the following pairs of points, find the change in the y co-ordinate and the corresponding change in the x co-ordinate. Hence, find the gradients of the lines passing through them:

(a) A(2, 3), B(5, 6) (b) C(5, 10), D(12, 20)

(c) E(–5, 6), F(2, 1) (d) G(4, 5), H(6, 5)

(e) I(8, 0), J(12, –6) (f) K(5, –2), L(6, 2)

(g) M(6, 3), N(–6, +2) (h) P(2, –5), Q(2, 3)

3. Find the gradients of the lines marked l_1, l_2 and l_3 in figure 4.7:

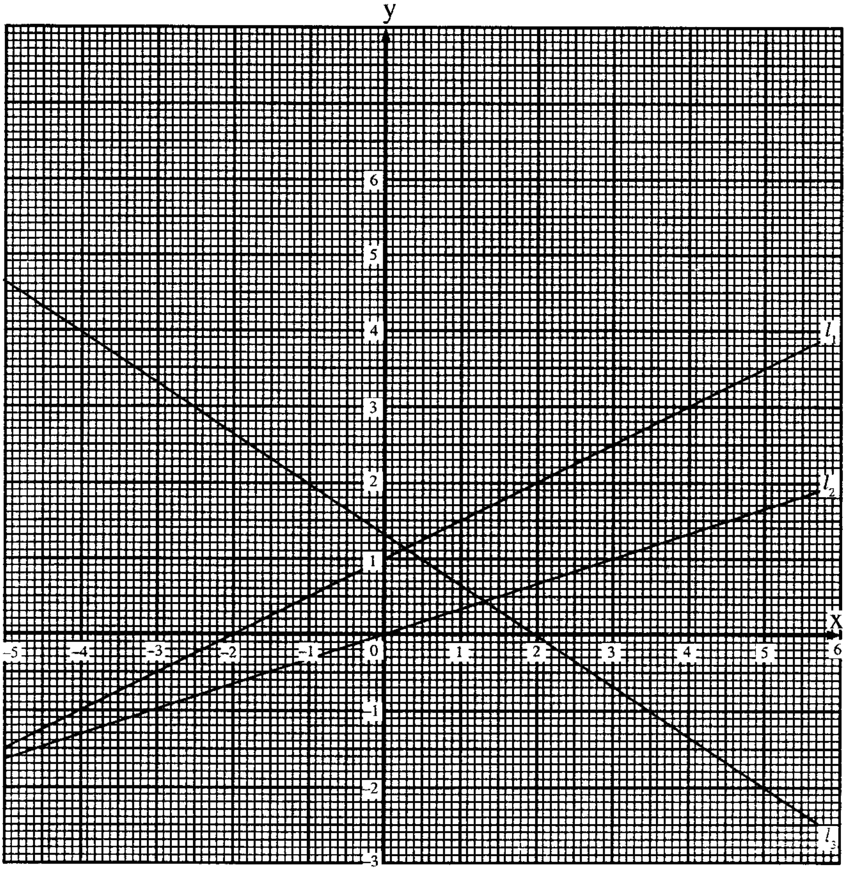


Fig. 4.7

4. Find the gradients of the lines passing through the following pairs of points:
   (a) A(3, 2), B(−1, 1) (b) C(7, 2), D(4, 3)
   (c) E(−1, −3), F(−2, −2) (d) G(0, 5), H(2, 5)
   (e) I($\frac{1}{4}$, $\frac{1}{3}$), J($\frac{1}{3}$, $\frac{1}{4}$) (f) K(0.5, 0.3), L(−0.2, −0.7)

5. Find the gradient of each of the following lines:
   (a) $y = \frac{1}{2}x + 3$ (b) $3y - 4x = 5$ (c) $y = -2x + 2$
   (d) $y + 2x - 3 = 0$ (e) $\frac{1}{3}x + \frac{1}{4}y = \frac{1}{12}$ (f) $y = -10$

6. In each of the following, the co-ordinates of one point on the line and the gradient of the line are given. Give the co-ordinates of two other points on the line:

(a) (1, 2); 5 (b) (2, 1); $-\frac{3}{2}$ (c) (–4, 5); 0

(d) (6, 1); $\frac{1}{3}$ (e) (0.5, 0.5); 0.5 (f) (0, 3); –0.75

7. For each of the following:
(i) draw two distinct lines with the given gradients.
(ii) give the co-ordinates of a pair of points on each line.

(a) 1 (b) 4 (c) –2 (d) 5

(e) $\frac{1}{2}$ (f) –4 (g) 0 (h) undefined

4.2: Equation of a Line

Given two Points

Example 4

Find the equation of the line through the points A(1, 3) and B(2, 8).

Solution

The gradient of the required line is $\frac{8-3}{2-1} = 5$

Take any point P(x, y) on the line.
Using the points P and A, the gradient is $\frac{y-3}{x-1}$

Therefore, $\frac{y-3}{x-1} = 5$

This simplifies to $y = 5x - 2$.
Find the equation of the line using the points P and B.

Give the Gradient and One Point on the Line

Example 5

Determine the equation of a line with gradient 3, passing through the point (1, 5).

Solution

Let the line pass through a general point (x, y). The gradient of the line is $\frac{y-5}{x-1} = 3$. This equation reduces to $y = 3x + 2$.

4.3: Linear Equation y = mx + c

We can express linear equations in the form $y = mx + c$. For example,

$4x + 3y = -8$ is equivalent to $y = -\frac{4}{3}x - \frac{8}{3}$

In this case, $m = -\frac{4}{3}$ and $c = -\frac{8}{3}$.

The constants m and c are significant in the geometry of a straight line. Consider the three lines l_1, l_2 and l_3 whose equations are $2y = x + 6$, $3y = 4x - 5$ and $2y = 7 - 5x$ respectively. Their graphs are given in figure 4.8.

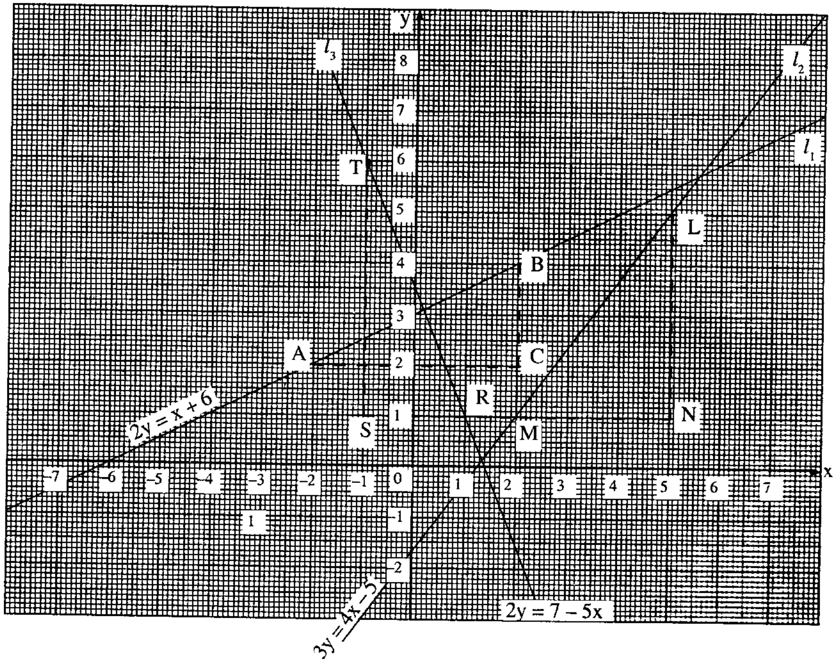


Fig. 4.8

The gradient of l_1 is $\frac{BC}{AC} = \frac{2}{4} = \frac{1}{2}$

The gradient of l_2 is $\frac{LN}{MN} = \frac{4}{3}$

The gradient of l_3 is $\frac{TS}{SR} = -\frac{5}{2}$

Re-write each of the above equations in the form $y = mx + c$, where m and c are constants and compare the value of m with that of the gradient. These are shown in the table below.

	Equation of line	*The form* $y = mx + c$	*m*	*c*	*Gradient*
l_1	$2y = x + 6$	$y = \frac{1}{2}x + 3$	$\frac{1}{2}$	3	$\frac{1}{2}$
l_2	$3y = 4x - 5$	$y = \frac{4}{3}x - \frac{5}{3}$	$\frac{4}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$
l_3	$2y = 7 - 5x$	$y = -\frac{5}{2}x + \frac{7}{2}$	$-\frac{5}{2}$	$\frac{7}{2}$	$-\frac{5}{2}$

In each case, we note that the gradient equals the corresponding value of m. If we put the equation of a straight line in the form $y = mx + c$, we can easily tell the gradient of the line. The gradient is equal to m.

Example 6

Find the gradient of the line whose equation is $3y - 6x + 7 = 0$

Solution

Write the equation in the form $y = mx + c$.

$3y = 6x - 7$

$y = 2x - \frac{7}{3}$

Here, $m = 2$ and so the gradient is 2.

The y-intercept

Consider the following equations of straight lines:

(i) $y = 3x + 2$

(ii) $y = -5x + 3$

(iii) $y = \frac{5}{2}x - 4$

For each of the lines:

(i) find the value of y when $x = 0$.

(ii) give the co-ordinates of the point where the lines cross the y axis.

You will notice that in general, the graph of $y = mx + c$ cuts the y axis at $(0, c)$. The number c is called the **y-intercept**. The y-intercepts of the above equations are 2, 3 and –4 respectively.

Note:

If a line passes through points $(a, 0)$ and $(0, b)$, then a and b are called the x-intercept and y-intercept respectively. The equation of such a line is

$\frac{x}{a} + \frac{y}{b} = 1.$

Example 7

Find the equation of a line whose x-intercept is –3 and y-intercept is 6.

Solution

Points (–3, 0) and (0, 6) lie on the line.

Therefore, the gradient of the line $= \dfrac{6-0}{0-(-3)}$

$= \frac{6}{3}$

$= 2$

In $y = mx + c$, m is the gradient of the line and c is the y-intercept.
In the above case, $m = 2$ and $c = 6$.
Therefore, the equation of the line is $y = 2x + 6$.

Use the general form $\frac{x}{a} + \frac{y}{b} = 1$ to find the equation.

Exercise 4.2

1. For each of the following straight lines, determine the gradient and the y-intercept. Do not draw the line:

   (a) $3y = 7x$ (b) $2y = 6x + 1$ (c) $7 - 2x = 4y$

   (d) $3y = 7$ (e) $2y - 3x + 4 = 0$ (f) $3(2x - 1) = 5y$

   (g) $y + 3x + 7 = 0$ (h) $5x - 3y + 6 = 0$ (i) $\frac{3}{2}y - 15 = \frac{2}{3}x$

   (j) $2(x + y) = 4$ (k) $\frac{1}{3}x + \frac{2}{5}y + \frac{1}{6} = 0$ (l) $-10(x + 3) = 0.5y$

   (m) $ax + by + c = 0$

2. Find the equations of lines with the given gradients and passing through the given points:

   (a) 4; (2, 5) (b) $\frac{3}{4}$; (–1, 3) (c) –2; (7, 2)

   (d) $-\frac{1}{3}$; (6, 2) (e) 0; (–3, –5) (f) $-\frac{3}{2}$; (0, 7)

   (g) m; (1, 2) (h) m; (a, b)

3. Write the equations of the lines l_1, l_2, l_3, l_4 shown in figure 4.9 in the form $y = mx + c$.

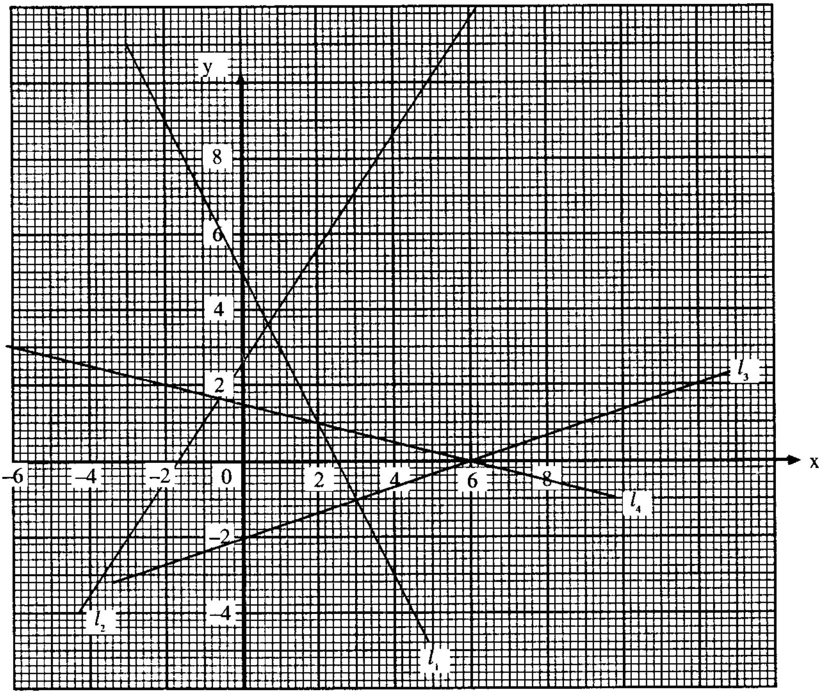


Fig. 4.9

4. Find the equation of the line passing through the given points:

(a) (0, 0) and (1. 3)
(b) (0, –4) and (1, 2)
(c) (0, 4) and (–1, –2)
(d) (1, 0) and (, 1)
(e) (3, 7) and (5, 7)
(f) (–1, 7) and (3, 3)
(g) (11, 1) and (14, 4)
(h) (5, –2.5) and (3.5, –2)
(j) (a, b) and (c, d)
(k) (x_1, y_1) and (x_2, y_2)

5. Find the co-ordinates of the point where each of the following lines cuts the x-axis:

(a) $y = 7x - 3$
(b) $y = -(3x + 2)$
(c) $y = \frac{1}{3}x + 4$
(d) $y = 0.5 - 0.8x$
(e) $y = mx + c \ (m \neq 0)$
(f) $ax + by + c = 0 \ (a \neq 0)$

6. The x and y-intercepts of a line are given below. Determine the equation of the line in each case:

	x-intercept	*y-intercept*
(a)	–2	–2
(b)	–3	4
(c)	5	–1
(d)	3	4
(e)	a	b ($a \neq 0$, $b \neq 0$)

7. The equation of the base of an isosceles triangle ABC is $y = -2$ and the equation of one of its sides is $y + 2x = 4$. If the co-ordinates of A are (–1, 6), find the co-ordinates of B and C. Hence, find the equation of the remaining side.

4.4: The Graph of a Straight Line

Example 8

Draw the graph of a line passing through (0, –4) and has a gradient of 2.

Solution

The equation of the line is;

$$\frac{y+4}{x} = 2$$

$$y + 4 = 2x$$
$$y = 2x - 4$$

The x-intercept is 2, and the y-intercept is –4. The line cuts the x-axis at (2, 0), and the y-axis at (0, –4).

The two points can then be plotted on cartesian plane and joined as shown in figure 4.10.

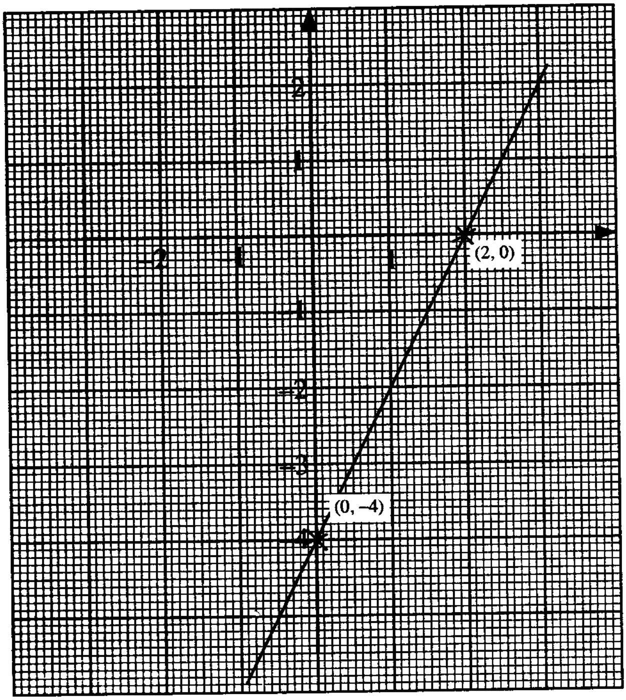


Fig. 4.10

Exercise 4.3

1. Draw the lines passing through the given points and having the given gradients:

   (a) (0, 3); 3 (b) (0, 2); 5 (c) (4, 3); 2

2. Draw the graph of the line passing through:

   (a) (5, 0) and gradient is 2 (b) (3, 0); g = 5 (c) (2, 0); $g = \frac{1}{3}$

3. Draw graphs of the lines represented by the following equations using the x and y-intercepts:

   (a) $y = \frac{1}{2}x + 3$ (b) $3y - 4x = 5$

   (c) $y + 2x - 3 = 0$ (d) $y = -2x + 2$

4.5: Perpendicular Lines

In figure 4.11, there are four pairs of perpendicular lines. List them:

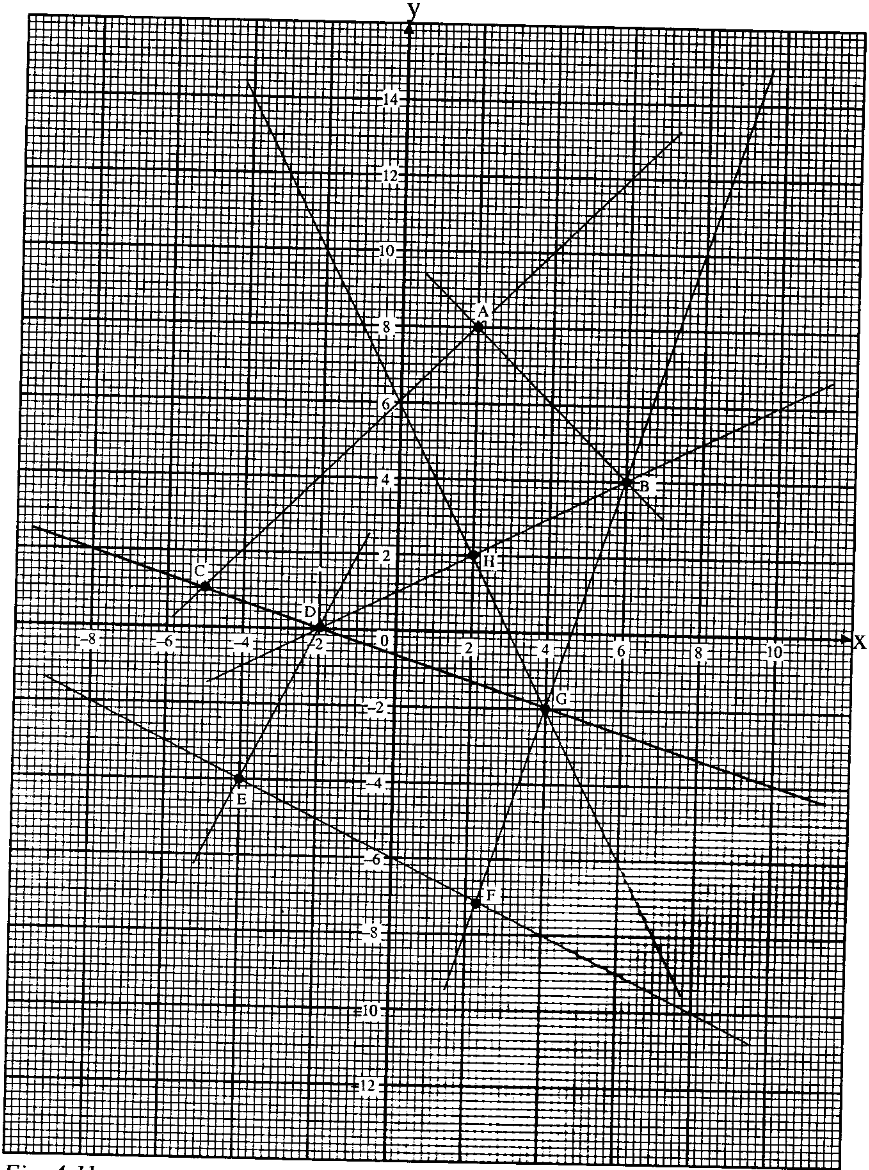


Fig. 4.11

Use the results of figure 4.11 to complete the table below:

Table 1.1

	Gradient		*Product of gradients*
	Line 1 (m_1)	*Line 2* (m_2)	$m_1 \times m_2$
1st pair			
2nd pair			
3rd pair			
4th pair			

You should notice from your table that $m_1 \times m_2 = -1$.

If the product of gradients of two lines is equal to –1, then the two lines are perpendicular to each other.

Example 9

Which of the following lines is perpendicular to the line $y = 4x + 2$?

(a) l_1: $y = \frac{1}{2} - \frac{1}{4}x$ (b) l_2: $y + 4x = 6$

Solution

(a) $l_1: y = \frac{1}{2} - \frac{1}{4}x$ can be re-written as $y = -\frac{1}{4}x + \frac{1}{2}$ (the gradient y-intercept form)

The gradient is $-\frac{1}{4}$

The gradient of $y = 4x + 2$ is 4.

The product of the two gradients is $-\frac{1}{4} \times 4 = -1$

Therefore, the two lines are perpendicular to each other.

(b) l_2: $y + 4x = 6$ can be re-written as $y = -4x + 6$

The gradient is -4.

The product of the gradient of l_2 and the line $y = 4x + 2$ is;

$-4 \times 4 = -16$.

Therefore, the two lines are not perpendicular to each other.

Exercise 4.4

1. Without drawing, determine which of the following pairs of lines are perpendicular:

(a) $y = 2x + 7$, $y = -\frac{1}{2}x + 3$ (b) $y = \frac{1}{3}x + 1$, $y = -3x - 2$ (c) $y = 2x + 7$, $y = -2x + 5$

(d) $y = 5x + 1$
$y = -\frac{1}{5}x + 2$

(e) $y = \frac{2}{3}x - 1$
$y = \frac{3}{2}x - 4$

(f) $y = \frac{2}{3}x + 4$
$y = -\frac{3}{2}x + 4$

(g) $y = \frac{7}{2}x + 2$
$y = \frac{2}{7}x - \frac{1}{2}$

(h) $y = -\frac{3}{4}x - 2$
$y = -\frac{4}{3}x + 5$

2. Determine the equations of the lines perpendicular to the given lines and passing through the given points:
(a) $y - 5x + 3 = 0$; (3, 2)
(b) $y = 8 - 7x$; (–3, –4)
(c) $y + 3x + 5 = 0$; (0.25, –0.75)
(d) $y + x = 17$; (–4, 2)
(e) $y = 17$; (–2, –1)
(f) $y = mx + c$; (a, b).

3. A triangle has vertices A(2, 5), B(1, –2) and C(–5, 1). Determine:
(a) the equation of the line BC.
(b) the equation of the perpendicular line from A to BC.

4. ABCD is a rhombus. Three of its vertices are A(1, 2), B(4, 6) and C(4, 11). Find the equations of its diagonals and the co-ordinates of vertex D.

5. The point D(–2, 5) is one of the vertices of a square ABCD. The equations of the lines AB and AC are $y = \frac{1}{3}x - 1$ and $y = 2x - 1$ respectively. Determine the equations of sides DA, DB and DC. Hence, find the co-ordinates of the remaining three vertices.

6. ABCD is a rectangle with the centre at the origin. A is the point (5, 0). Points B and C lie on the line $2y = x + 5$. Determine the co-ordinates of the other vertices.

4.6: Parallel Lines

Draw two lines with the following gradients:

(a) $\frac{1}{5}$ (b) 2 (c) –2 (d) 5

What do you notice about the lines?
You should have noticed that lines with the same gradient are parallel.
In figure 4.12, state the gradients of the lines:

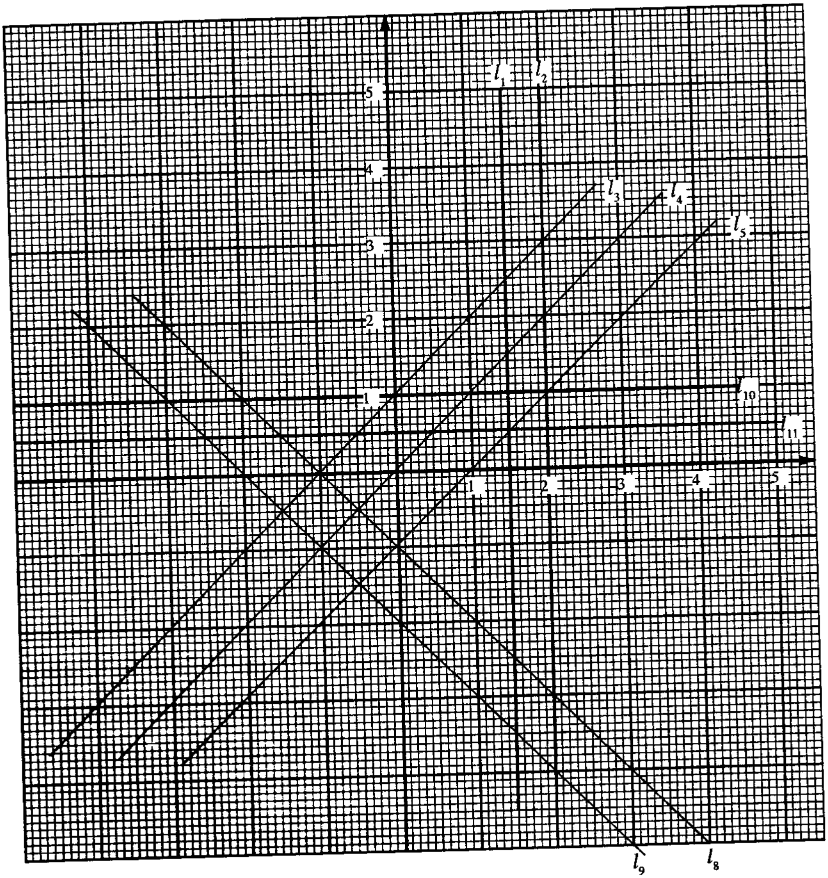


Fig. 4.12

Exercise 4.5

1. Determine which of the following pairs of straight lines are parallel. Do not draw the lines:

(a) $y = \frac{1}{2}x + 7$

$y = \frac{1}{2}x - 20$

(b) $\frac{3}{2}x + \frac{2}{3}y + \frac{3}{2} = 0$

$\frac{2}{3}x + \frac{3}{2}y + \frac{2}{3} = 0$

(c) $2x + 3y - 8 = 0$

$21 - 4x - 6y = 0$

(d) $5x + 6y - 4 = 0$

$y = \frac{6}{5}x + \frac{2}{3}$

(e) $2y - 7x - 8 = 0$

$4y + 17 = 14x$

(f) $y = -\frac{4}{5}x + 13$ and the line through $(-1, -3)$ and $(-3.5, -11)$

(g) $6(x - 3y) + 7 = 0$ and the line through the origin having gradient -3.
(h) A line through (0, 7) and (1, 11) and another line through (1, 11.8) and (2, 14.6)
(i) A line through (1, 4) and (−1, −1) and another line through (99, 103) and (51, 7)

2. In each of the following, find the equation of the line through the given point and parallel to the given line:

(a) $(0, 0); y = -\frac{2}{7}x + 1$
(b) $(3.5, 0); x + y = 10$
(c) $(5, 2); 5y - 2x - 115 = 0$
(d) $(-3, 5); 7y = 3x$
(e) $(-\frac{7}{3}, \frac{3}{4}); 2(y - 2x) = 1.1$
(f) $(0, -3); 2x + y = -3$
(g) $(3\frac{1}{7}, -1\frac{1}{7}); 15(1-x) = 22y$
(h) $(-3, 4); x = 101$
(i) $(2, 3); y = 0$

3. The equations of two sides of a parallelogram are $y = -2$ and $y = x - 2$. If one of the vertices of the parallelogram has co-ordinates (7, 5), find the equations of the other two sides and the remaining vertices.

4. In a rectangle ABCD, the equations of the line AB is $3y = x + 6$. The x co-ordinate of A is −3. The line AD is parallel to the line $y + 3x = 7$. If C has the co-ordinates (2, 6), determine:
(a) the equation of lines AD, BC and CD.
(b) the co-ordinates of A, B and D.
(c) the equations of the diagonals.

Chapter Five

REFLECTION AND CONGRUENCE

5.1: Symmetry

Figure 5.1 shows a rectangle ABCD. X and Y are the midpoints of AD and BC respectively, while P and Q are the midpoints of AB and DC respectively.

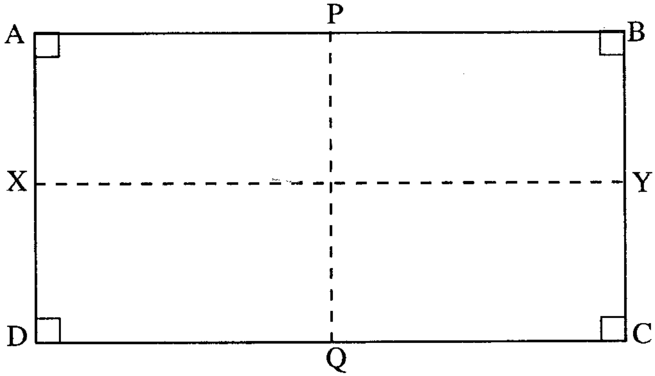


Fig 5.1

Trace the figure and fold the tracing along the line XY. You will notice that ABYX fits exactly onto DCYX. Similarly, fold the tracing along PQ. You will also notice that DAPQ fits exactly onto PQCB.

The lines of fold such as XY and PQ which divide a plane figure into two identical parts are called **lines of symmetry.**

Fold the tracing along AC and BD. You should notice that the diagonals of a rectangle are not lines of symmetry.

From the above, we notice that a rectangle has two lines of symmetry.

(a) How many lines of symmetry do the following shapes have:
    (i) a square   (ii) a kite   (iii) a circle

(b) Name other shapes which have lines of symmetry.

If an orange is cut into two identical parts, the cut is called the **plane of symmetry**. How many planes of symmetry does an orange have? Give other examples in nature that have planes of symmetry.

Exercise 5.1

1. Find how many lines of symmetry each of the following shapes has:

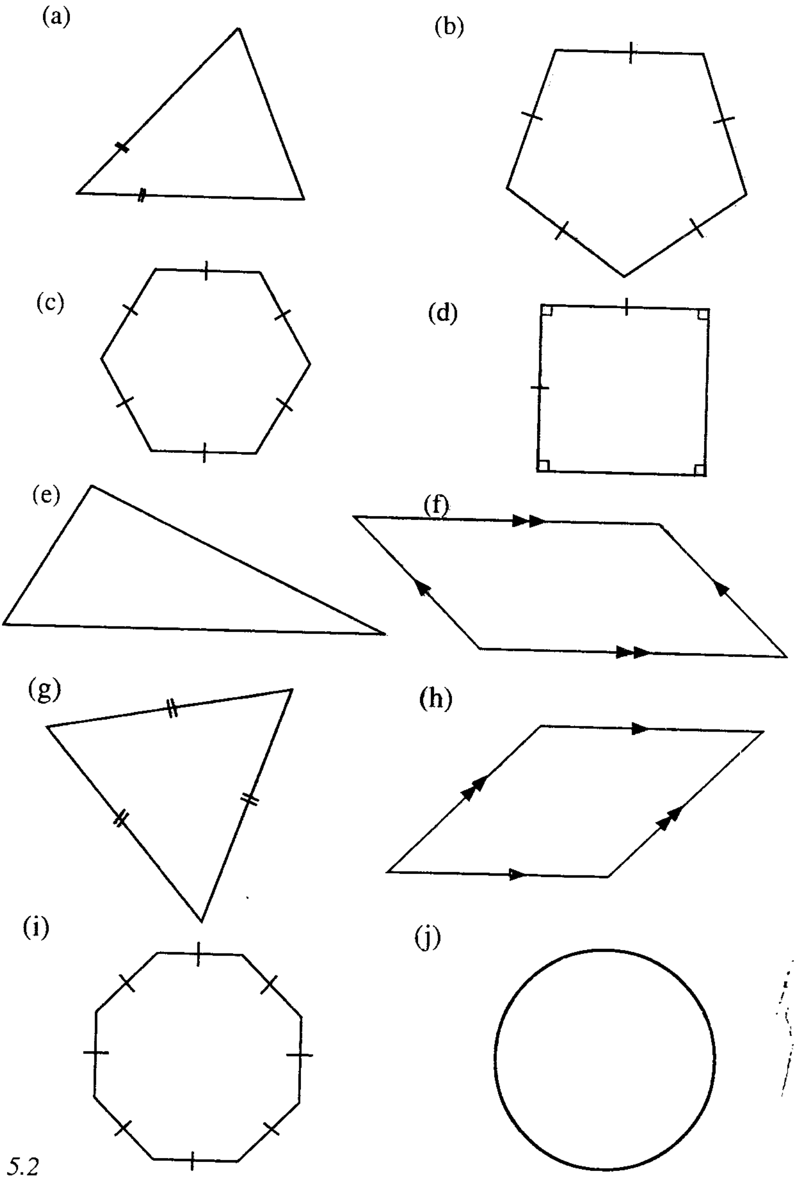


Fig. 5.2

2. Which letters of the alphabet have lines of symmetry?
3. State the number of planes of symmetry the following solids have:
   (a) A square-based pyramid.
   (b) A spherical ball.
   (c) A cuboid.
   (d) A cylindrical piece of wood.
   (e) A cone.
   (f) A prism whose cross-section is:
      (i) an isosceles triangle.
      (ii) an equilateral triangle.
      (iii) a scalene triangle.
4. Name some parts of your body that are mirror images of each other.
5. How many planes of symmetry has the solid in figure 5.3?

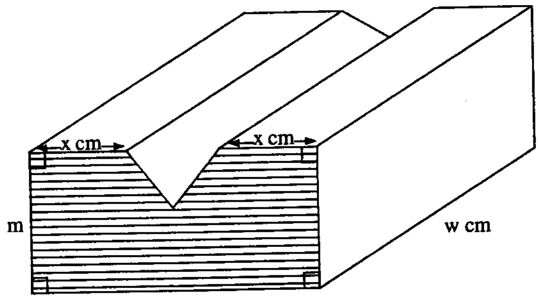


Fig. 5.3

5.2: Reflection

Consider figure 5.4:

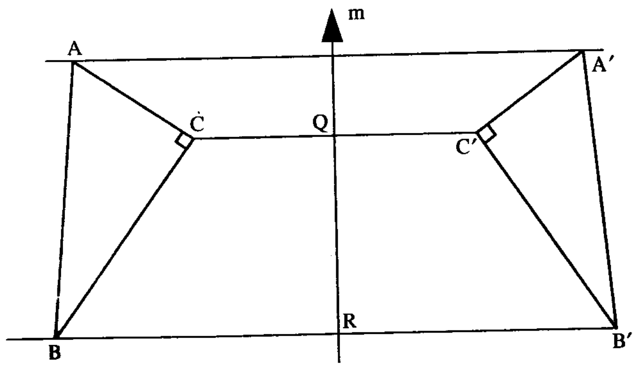


Fig. 5.4

In the figure, triangle $A'B'C'$ is the image of triangle ABC under a reflection in the line m.

A is mapped onto A'

B is mapped onto B'

C is mapped onto C'

Now, measure AP and PA', BR and RB' and CQ and QC'. Measure also the angles between the mirror line and lines AA', BB' and CC'. What do you notice?

You should notice that:

(i) a point on the object and a corresponding point on the image are equidistant from the mirror line.

(ii) the line joining a point and its image is perpendicular to the mirror line. You should also notice that the object and its image have the same shape and size.

Example 1

Draw the image of triangle ABC under a reflection in the mirror line M.

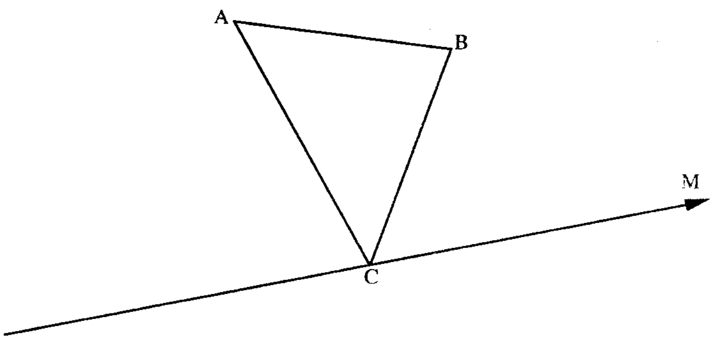


Fig. 5.5

Solution

To obtain the image of A, drop a perpendicular from A to the mirror line and produce it. Mark off A', the image of A equidistant from the mirror line as A. Similarly, obtain B', the image of B. Since C is on the mirror line, it is mapped onto itself. Join A' to B', B' to C' and C' to A' to obtain triangle $A'B'C'$, the image of triangle ABC.

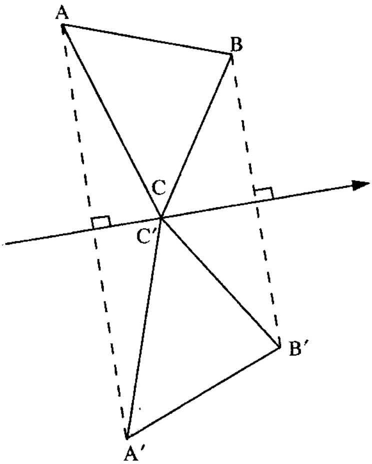


Fig. 5.6

A mirror line is a line of symmetry between an object and its image.

Measure the angles between the mirror line and the lines:

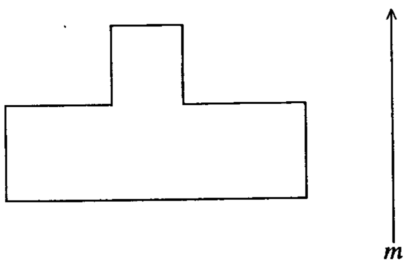
(a) (i) B′C (b) (i) A′C
(ii) BC (ii) AC

You should notice that the mirror line bisects the angles B′CB and A′CA. In general, a mirror line is an angle bisector.

Exercise 5.2

1. In each of the diagrams in figure 5.7, m is the mirror line. Make a tracing of each and draw their images under a reflection in the mirror line.

(a)



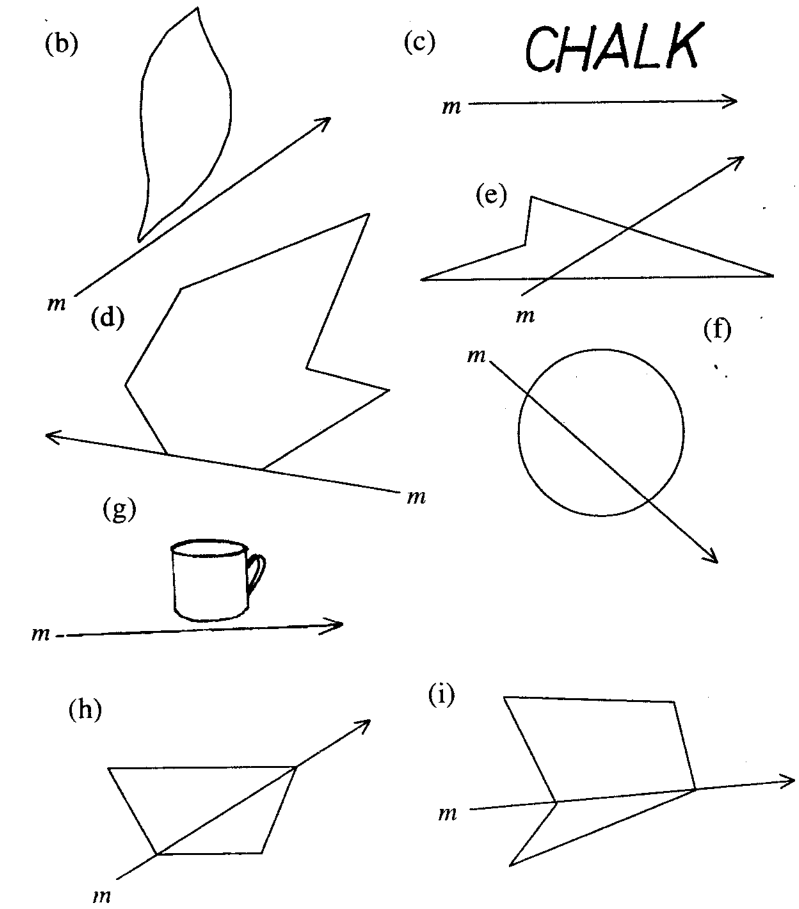


Fig. 5.7

2. An object line AB, 4 cm long, makes an angle of 33° with the mirror line at point B. Draw the image line and find the angle between $A'B$ and the mirror line.
3. The angle between an image line and its object line is 80^{0}. Draw a diagram to show the position of the mirror line.
4. Figure 5.8 shows an object and its image after reflection. Trace them and indicate the mirror line.

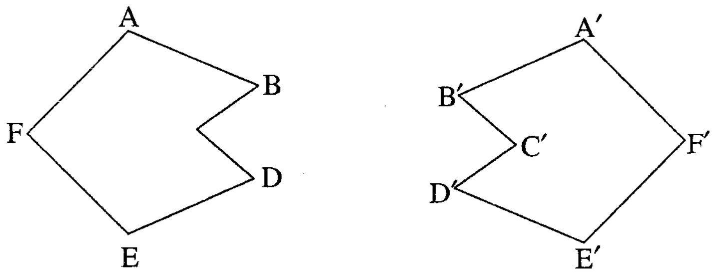


Fig. 5.8

5. The vertices of a quadrilateral ABCD are A(4.5, 1.5), B(7.5, 3), C(5, 3.5) and D(4.5, 2.5). Draw on the same axes the quadrilateral and its image after a reflection in:
   (a) the x-axis. (b) the line $y = x$.
   (c) the y-axis . (d) the line $y = -x$.
   In each case, determine the co-ordinates of the vertices of the image.
6. If A(2, –7), B(2, –2) and C(7, –2) are the vertices of a triangle, find the image of the triangle under a reflection in:
   (a) the line $y = 2.5$. (b) the line $y = -2.5$,
   (c) the line $x = 3.5$. (d) the line $x = -3.5$
7. A(–5, –2), B(–2, –5) and C(–12, –5) are vertices of a triangle. Find the image of the triangle when it is reflected in the:
   (a) y-axis followed by a reflection in the $y = -x$.
   (b) x-axis followed by a reflection in the line $y = x$.
8. The vertices of a polygon ABCDEF are A(–4, 6), B(–3, 2), C(–7, 1), D(–7, 4), E(–8, 5) and (F(–7, 6). Find the final image of the polygon under the reflection in the line:
   (a) $y = x$, followed by a reflection in $y = -x$.
   (b) $y = x$, followed by a reflection in y axis, and finally followed by one in the line $y = -x$.
9. A circle with centre A(7, 4) and radius 3 units is reflected in the line $x = 4.5$, followed by a reflection in the y-axis. Draw the final image.
10. In figure 5.9, the letter W is first reflected in the line $y = x$ followed by a reflection in the y-axis. Determine the co-ordinates of its final image.

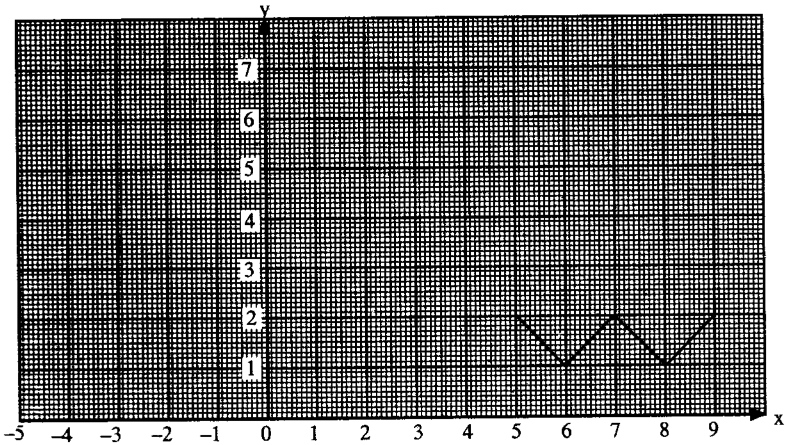


Fig. 5.9

11. A letter T has vertices at (3, 8), (4, 8), (4, 6) and (5, 8). It is reflected in the x-axis. Draw the image of T.

12. A letter N has vertices at (2, 6), (3, 6), (3, 8) and (4, 8). It is reflected in the y-axis followed by a reflection in the line y = 2. Determine the co-ordinates of the final image.

13. The co-ordinates of the vertices of the letter V are (8, 4), (9, 2) and (10, 4). If the letter is reflected in the line x = −2 followed by reflection in the line y = x, determine:
    (a) the co-ordinates of the first image.
    (b) the co-ordinates of the final image.

14. Figure 5.10 shows a flag made from a quadrant of a circle ABC and fixed to a pole BCD. The diagram is reflected in the line y = 8, followed by a reflection in the line y = x. Draw the image of the flag:
    (a) under the first reflection.
    (b) under the final reflection.

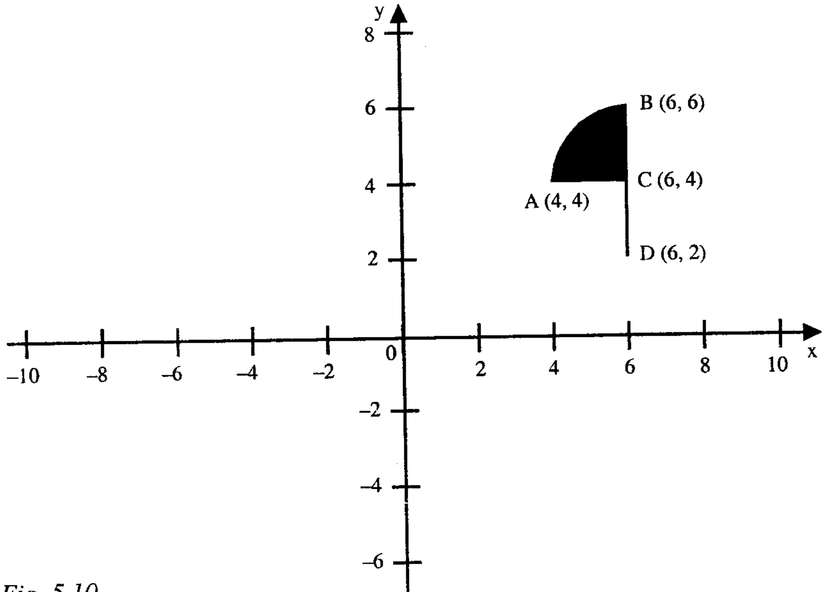


Fig. 5.10

15. In figure 5.11 the object is reflected in line x = 6 followed by a reflection in line y = – x. Find the co-ordinates of :
 (a) the image under the first reflection.
 (b) the image under the final reflection.

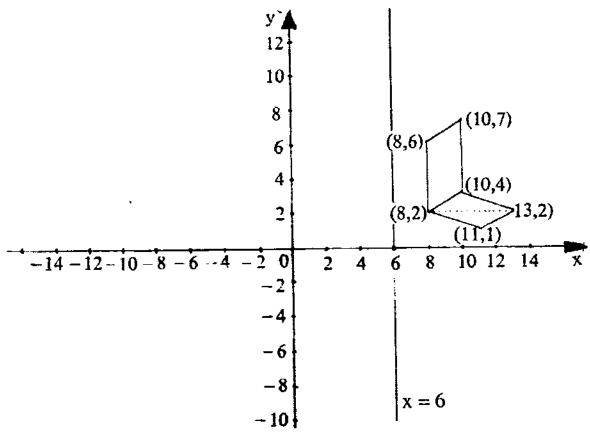


Fig. 5.11

16. Draw the pattern shown in figure 5.12 on a graph paper and show its image after a reflection in the line y = x:

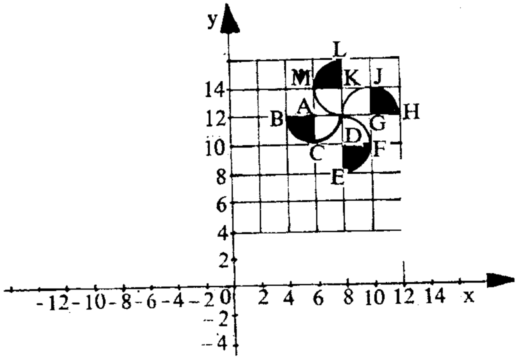


Fig. 5.12

17. The object in figure 5.13 is reflected in the line x = –4, followed by a reflection in the line y = x. Find the position of the image:
    (a) under the first reflection.
    (b) after the second reflection.

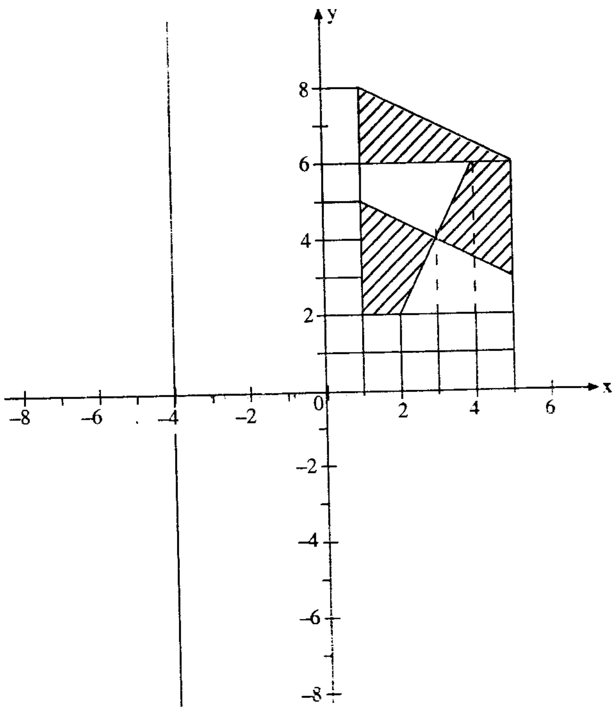


Fig. 5.13

18. Draw the image of the object in figure 5.14 under a reflection in the line PQ followed by a reflection in the line y = x:

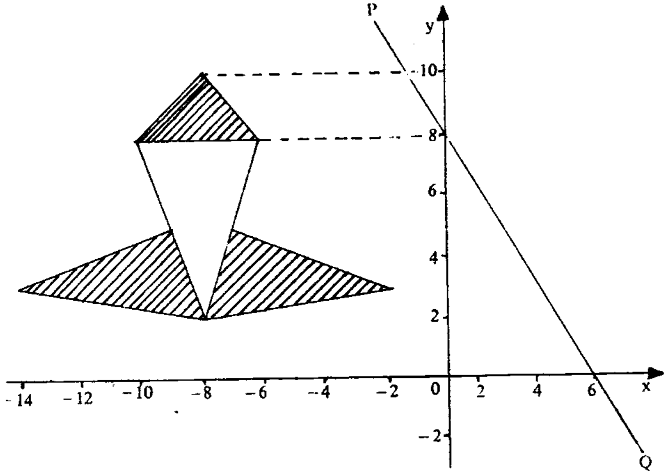


Fig. 5.14

5.3: Some Geometrical Deductions Using Reflection

(*a*) *Vertically opposite angles are equal*

The lines L_1 and L_2 intersect at O as shown in figure 5.15:

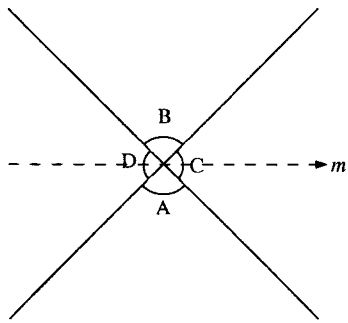


Fig. 5.15

The broken line showing the mirror line m bisects angle C and angle D, as shown in figure 5.16. Show that angle A equals angle B and angle C equals angle D.

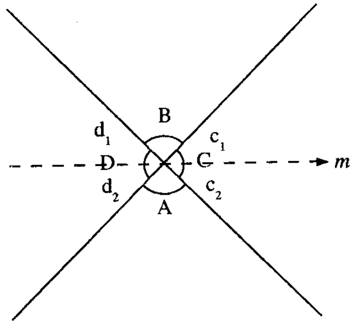


Fig. 5.16

Let $D = d_1 + d_2$ and $C = c_1 + c_2$ (reflection in m)

$c_1 + B + d_1 = 180^0$ (angles on a straight line).

$c_2 + A + d_2 = 180$ (angles on a straight line).

But $c_1 = c_2$ and $d_1 = d_2$

Therefore, $\angle A = \angle B$

Therefore, vertically opposite angles are equal.

$B + c_1 + c_2 = d_2 + d_1 + B$

So, $c_1 + c_2 = d_1 + d_2$, but $c_1 + c_2 = C$ and $d_1 + d_2 = D$

Therefore, $C = D$

Therefore, if two straight lines intersect, the vertically opposite angles are equal.

(*b*) *Base angles of an isosceles triangle are equal*

Figure 5.17 shows a right-angled triangle ABC and its image ABC' under a reflection in AF .

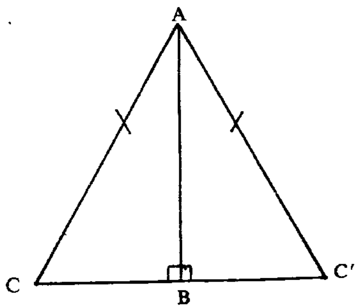


Fig. 5.17

By reflection, $AC = AC'$, $BC = BC'$ and $\angle C = \angle C'$
Triangle CAC' is therefore isosceles. Hence, by reflection, the base angles of an isosceles triangle are equal.

(c) *Angle sum of a triangle is 180^0*

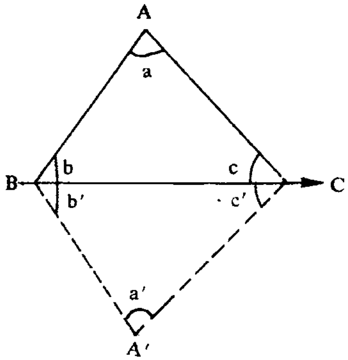


Fig. 5.18

Figure 5.18 shows two triangles, ABC and BCA′. BC is the mirror line.

By reflection, angle $a = a'$, $\angle b = \angle b'$ and $\angle c = \angle c'$.

But $\angle a + \angle b + \angle c + \angle a' + \angle b' + \angle c' = 360^0$ (angle sum of a quadrilateral).

Thus, $2a + 2b + 2c = 360^0$.

$2(a + b + c) = 360^0$

Therefore, $a + b + c = 180^0$.

Angle sum of a triangle is 180^0.

(d) *A kite and a rhombus*

(i) Figure 5.19 shows a kite ABCD in which AB = AD and BC = CD. AC is the mirr

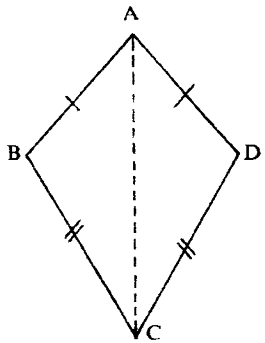


Fig. 5.19

Is there any other mirror line that can be drawn on the kite? Use the mirror line to show that the angle sum of the kite is 360°. Since B and D are images of each other then AC and BD meet at right angles.

(ii) Similarly, diagonals of a rhombus, which is a special kite, bisect at right angles. How many lines of symmetry does a rhombus have? Use reflection to show that the diagonals bisect the angles of a rhombus.

(e) *The mirror line as a perpendicular bisector*

A line equidistant from two given points L and M is the perpendicular bisector of LM.

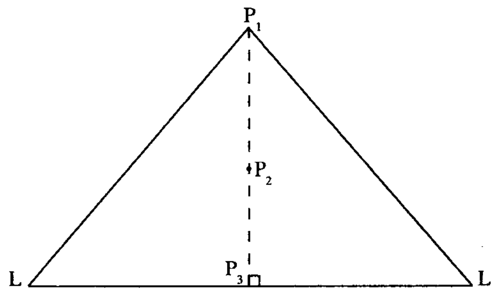


Fig. 5.20

Figure 5.20 shows two given points L and M. A point P is such that

PL = PM. Show that P lies on the perpendicular bisector of LM.

P_1, P_2 and P_3 show three possible positions of P. At P_3, P is on LM. Therefore,

$LP_3 = P_3M$. So. P_1P_3 is a bisector of LM.

Since $P_1L = P_1M$, ΔP_1LM is isosceles. Thus. the line P_1P_3 is perpendicular

bisector of LM. It is also a line of symmetry of ΔP_1LM, or the mediator.

Exercise 5.3

1. Line y = 2 intersects line x = 0. Write down the equations of two possible mirror lines under which the lines will be images of each other.
2. Write down the equation of the mediator of the lines joining each of the following pairs of points:
   (a) A(2, 8) and B(2, 10) (b) C(2, 8) and D(–2, 8)
   (c) E(–2, –3) and F(–4, –3) (d) G(2, –4) and H(3, –4)
   (e) I(5, 0) and J(0, 5)

3. Plot on a graph paper the points A(–2, –3), B(–4, –4) and C(–4, –2). Join all the points to form a triangle ABC.
   (a) Name the type of triangle formed.
   (b) Write down the equation of the mirror line if B and C are images of each other.

4. Four of the vertices of an irregular octagon, which is symmetrical about $y = x$, are $A(1, -\frac{1}{2})$, B(4, –3), C(4, 0) and D(3, 0). Find the co-ordinates of the remaining vertices. The octagon is reflected in the line $y = 0$. Find the equation of line of symmetry of the image.

5. Plot points A(0, 5), B(3, 9), C(3, 4) and D(0, 0).
   (a) What type of figure is quadrilateral ABCD?
   (b) Write down the equations of the two lines of symmetry.
   (c) Find the point of intersection of the lines of symmetry.

6. Plot points A(4, 1) and B(2, 5).
   (a) Calculate the distance AB.
   (b) Using mirror line $y = -x$, reflect line AB.
   (c) What do you notice about the distances AB and $A'B'$?

7. Plot points P(–3, 7), Q(–5, 3), R(–3, 1) and S(–1, 3).
   (a) What type of figure is PQRS?
   (b) What is the equation of the line of symmetry?
   (c) Reflect figure PQRS in the line $y = x$ and find the new positions of P, Q, R and S.
   (d) What is the equation of the line of symmetry of the new figure?

8. A point P lies on the perpendicular bisector of the line LM. Given the co-ordinate of L as (2, 1) and M as (6, 1):
   (a) write down the equation of the perpendicular bisector.
   (b) find the co-ordinates of the point where the perpendicular bisector cuts LM.

9. Use squared paper to plot the points U(1, 4), V(5, 4) and X(3, 1). UX and VX intersect at X. A point T is such that TX is the angle bisector of $\angle UVX$. Locate one possible point of T and find the equation of TX.

5.4: Congruence

In figure 5.21, ΔABC is reflected in the mirror line M to obtain $\Delta A'B'C'$.

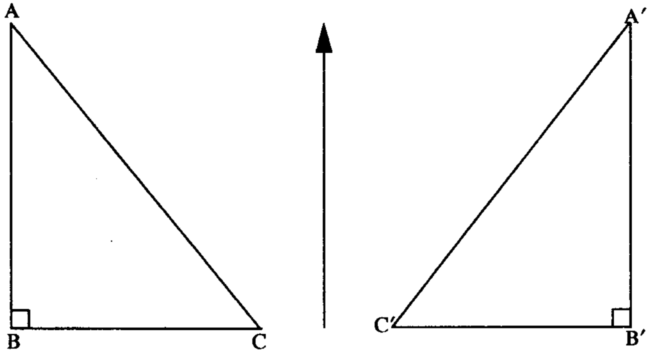


Fig. 5.21

Since $\Delta A'B'C'$ is a reflection of ΔABC, the two triangles:

(a) have the same shape.

(b) have the same size.

In general, figures which have the same shape and same size are said to be **congruent**. The symbol for congruency is $\equiv$. Thus, $\Delta ABC \equiv \Delta A'B'C'$.

Figure 5.22 shows two successive reflections. If we trace and slide ΔABC, it can fit onto $\Delta A''B''C''$ directly. In such a case, the congruency is said to be **direct**, i.e., the figures are **directly congruent**.

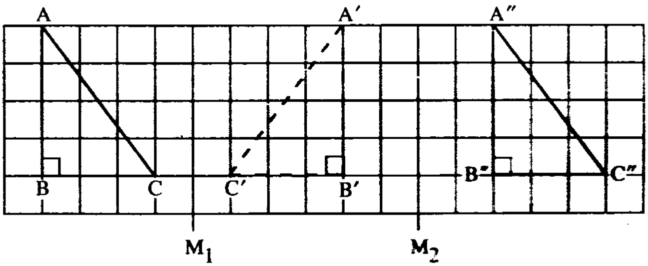


Fig. 5.22

Consider ΔABC and $\Delta A'B'C'$ in figure 5.22. In this case, ΔABC cannot fit onto $\Delta A'B'C'$ directly, but can fit when turned over. This congruence is called **opposite congruence (indirect congruence)**

Therefore: (i) triangles ABC and $A''B''C''$ are directly congruent.

(ii) triangles ABC and $A'B'C'$ are oppositely congruent.

Which of the following shapes in figure 5.23 are directly congruent? Which ones are oppositely conguent?

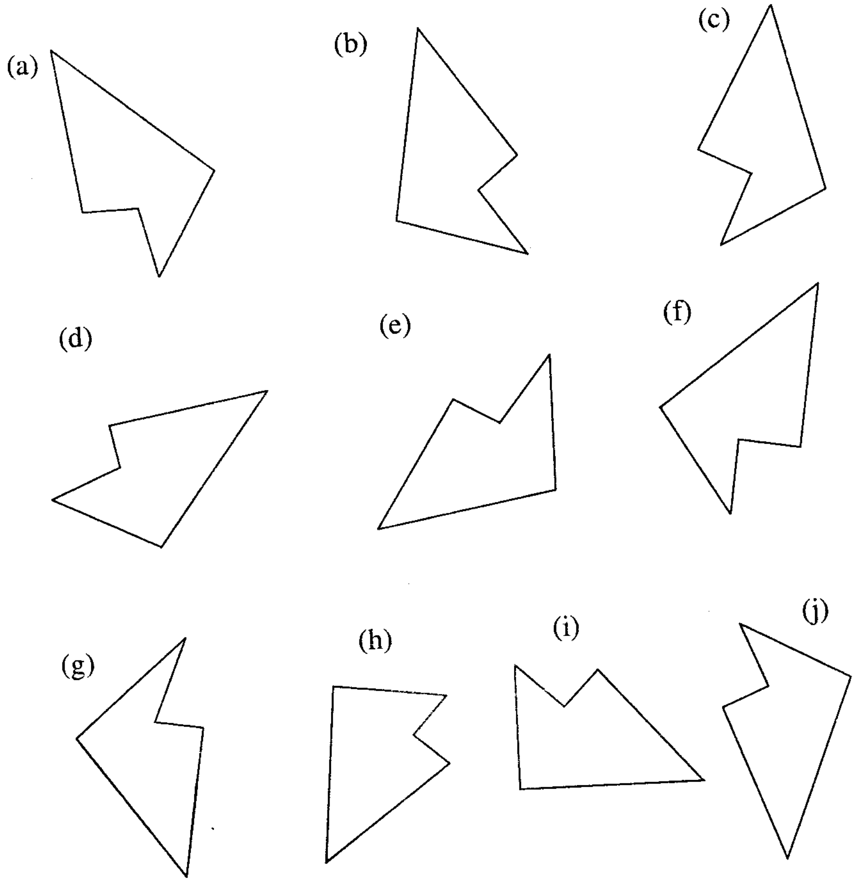


Fig. 5.23

Exercise 5.4

1. Which of the following figures are:
   (a) directly congruent?
   (b) oppositely congruent?

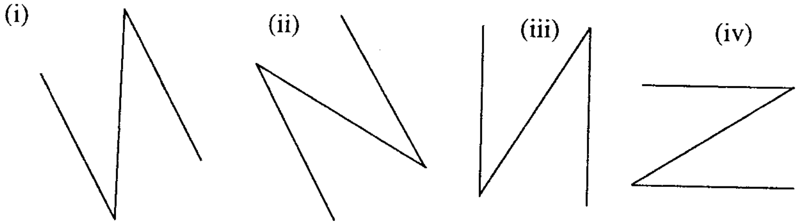


Fig. 5.24

2. In figure 5.25, write all pairs of triangles that are:
   (a) directly congruent.
   (b) oppositely congruent.

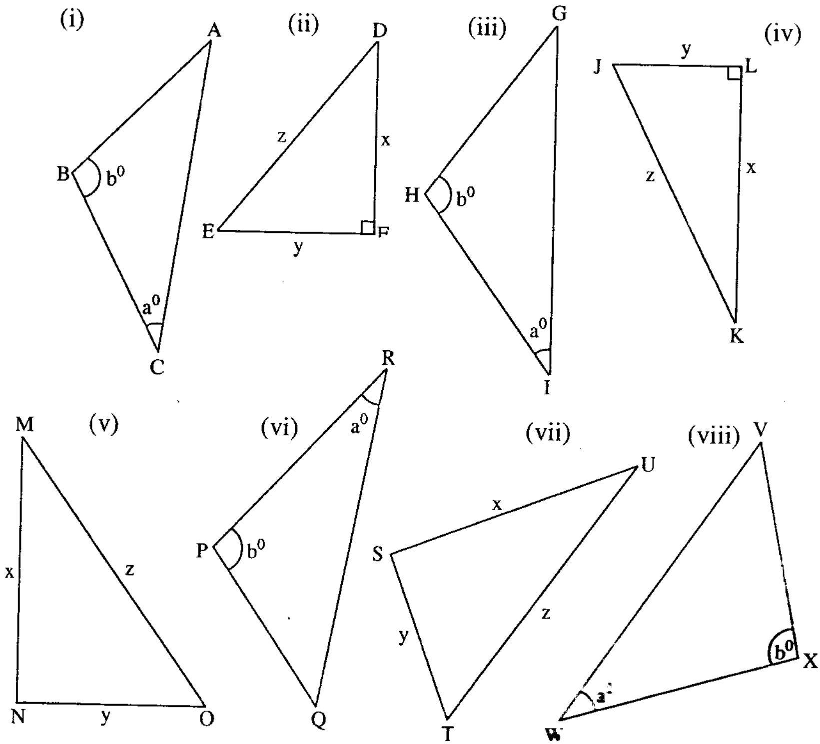


Fig. 5.25

3. What type of triangle would be both directly and oppositely congruent to its image after a reflection?

5.5: Congruent Triangles

Triangles which are congruent are named in such a way that the letters are written in the correct order so as to indicate the corresponding sides or the corresponding angles.

If ΔABC is congruent to ΔPQR, AB corresponds to PQ, BC to QR and CA to RP. Similarly, if $\angle ABC$ corresponds to $\angle PQR$, $\angle BCA$ corresponds to $\angle QRP$ and $\angle CAB$ corresponds to $\angle RPQ$.

Two or more triangles are congruent if they satisfy any of the following conditions:

(i) Three sides of one triangle are equal to the three corresponding sides of the other triangle (SSS).

(ii) Two sides and the **included** angle of one triangle are equal to the corresponding two sides and the **included** angle of the other triangle (SAS).

(iii) Two angles and a side of one triangle are equal to the corresponding two angles and a side of the other triangle (AAS or ASA).

(iv) A right angle, a hypotenuse and another side of one triangle are equal to the corresponding right-angle, a hypotenuse and another side of the other triangle (RHS).

The following examples illustrate the four conditions:

(i)

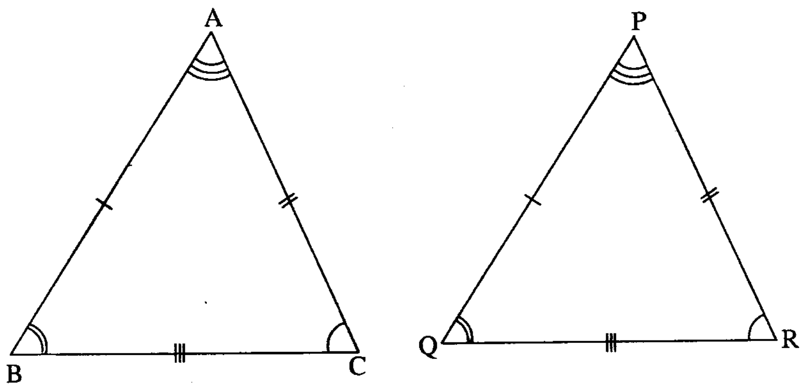


Fig. 5.26

In figure 5.26, Δs ABC and PQR are congruent. All the three corresponding sides are equal (SSS). AB = PQ, BC = QR and CA = RP.

The corresponding angles are equal since the corresponding sides are equal, e.g., $\angle ABC = \angle PQR$. Name other angles that are equal.

(ii)

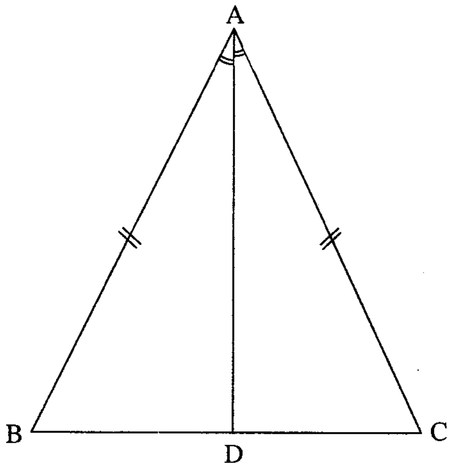


Fig. 5.27

Given the isosceles triangle ABC (figure 5.27), AD is a bisector of $\angle BAC$ and ΔABD is congruent to ΔACD, because line AD is common.

$\angle BAD = \angle DAC$ (given) and line AB = AC (given). Two sides and an included angle are equal (SAS).

(iii)

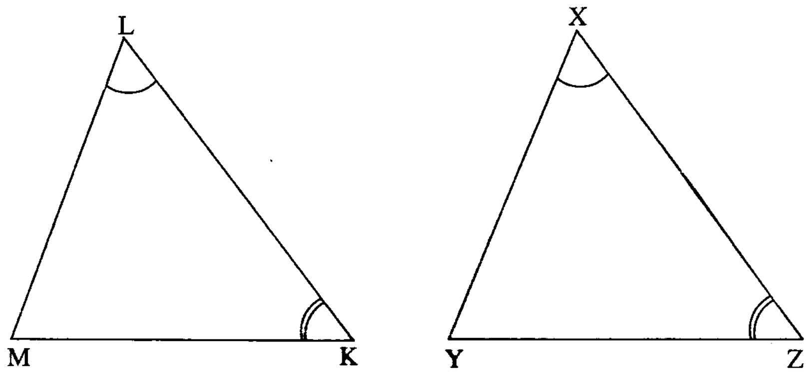


Fig. 5.28

Figure 5.28 shows two triangles in which LK = XZ, ∠MLK = ∠YXZ and ∠LKM = ∠XZY. This implies that ∠LMK = ∠XYZ, LM = XY and MK = YZ. The two triangles are, therefore, congruent.

Note:

In this case, the corresponding sides are between the pairs of the angles that are given (ASA).

At times, the given sides may not be between the given angles, as shown in figure 5.29. In the figure, ∠FEG = ∠SRT, ∠GFE = ∠TSR and FG = ST.

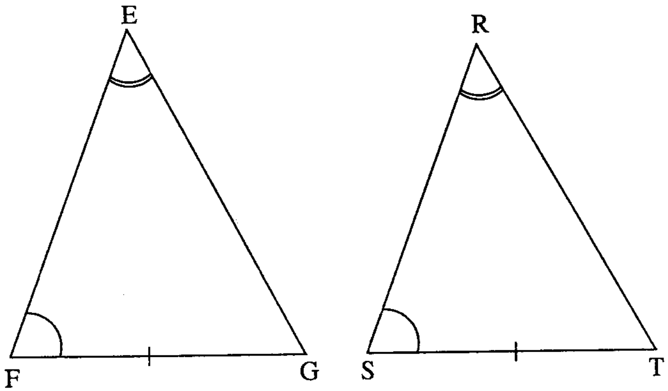


Fig. 5.29

Since two pairs of angles are equal, the third pair must also be equal. The two triangles are therefore congruent, provided the equal sides correspond (AAS or SAA).

(iv)

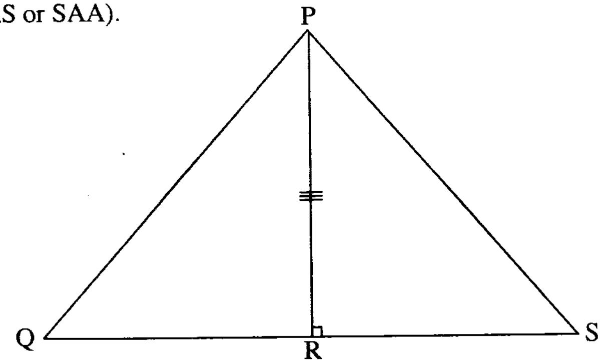


Fig. 5.30

Given the isosceles triangle PQS in figure 5.30, PQ = PS and line PR is perpendicular to the line QS. Triangles PQR and PSR are congruent. This is because:

- PR is a common line.
- $\angle PRQ = \angle PRS = 90^0$.
- PQ = PS (given).

Therefore, there is a right angle, a hypotenuse and another side equal (RHS). Note that in this case, the SAS test of congruence is not necessary.

(v) The ambiguous case

Consider a case where two sides and a non-included angle (which is not a right angle) are given. It is possible to draw two different triangles as shown in figure 5.31 (b) and (c) from the measurements of 5.31 (a).

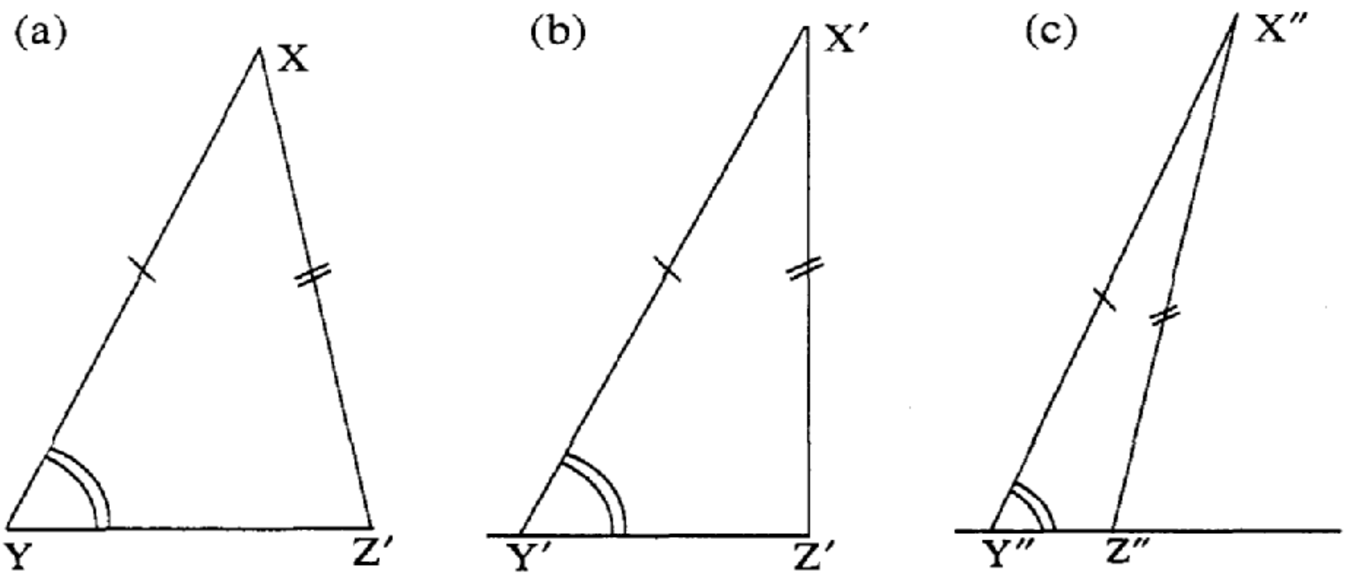


Fig. 5.31

(i) $\Delta X'Y'Z'$ **is congruent** to ΔXYZ because $XY = X'Y'$, $XZ = X'Z'$ and, $\angle XYZ = \angle X'Y'Z'$ implies $\angle XZY = \angle X'Z'Y'$

(ii) ΔXYZ **is not congruent** to the other possible triangle $X''Y''Z''$ since $\angle XZY \neq \angle X''Z''Y''$. However, the two angles are supplementary to each other. The fact that the given angle is not included makes the case ambiguous.

Exercise 5.5

1. In figure 5.32, PQ = PS and QR = RS. Show that ΔPQR and ΔPRS are congruent.

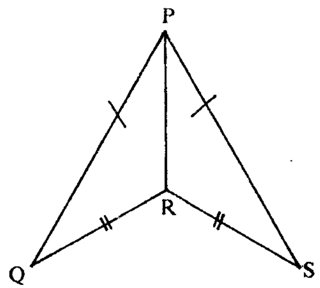


Fig. 5.32

2. In figure 5.33, points A, B and C are on the circumference of the circle with centre O. Show that ΔAOB and ΔAOC are congruent.

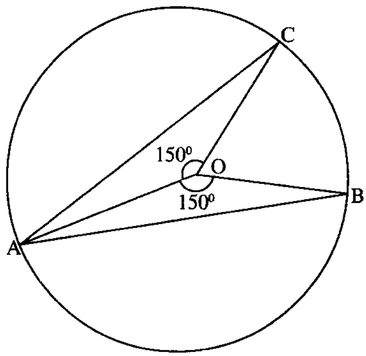


Fig. 5.33

3. In figure 5.34, ABCD is a parallelogram. Line BE is parallel to line FD.

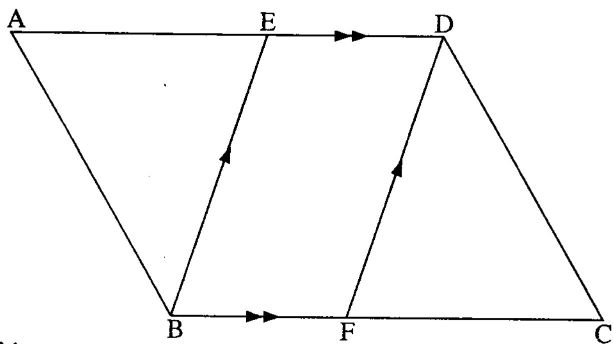


Fig. 5.34

Show that Δs ABE and CDF are congruent.

4. Two chords AB and AC of a circle are equal in length and the bisector of angle BAC meets the arc BC at D. Show that:
   (a) BD = DC.
   (b) AD is perpendicular to BC.
5. In figure 5.35, PS = QR = PX = XQ. Show that ΔPQS and ΔPQR are congruent.

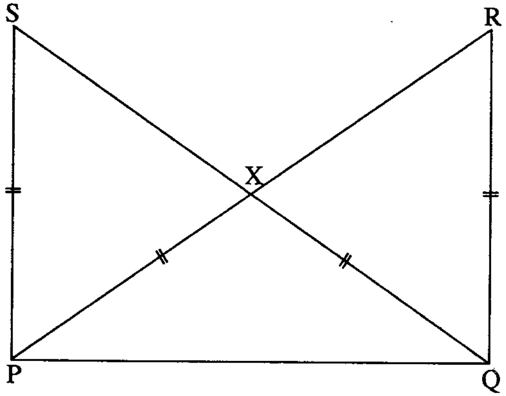


Fig. 5.35

In each of the numbers 6-13, construct the given triangle and determine whether it is an ambiguous case or not

6. ΔABC, in which BC = 8 cm, AB = 6.7 cm and $\angle ACB = 50^0$.
7. ΔPQR, in which QR = 8 cm, PQ = 6 cm and $\angle PRQ = 40^0$.
8. ΔXYZ, in which XZ = 12 cm, $\angle XYZ = 70^0$ and $\angle XZY = 50^0$.
9. ΔABC, in which BC = 8 cm, AC = 3.5 cm and $\angle ABC = 30^0$.
10. ΔABC, in which BC = 6 cm, AC = 5 cm and $\angle ABC = 50^0$.
11. ΔQRS, in which RS = 7 cm and the hypotenuse RQ = 13.3 cm.
12. ΔPQR, in which PR = 6 cm, QR = 4.6 cm and $\angle PQR = 130^0$.
13 ΔABC, in which $\angle ABC = 140^0$. AC = 8 cm and BC = 5 cm.
14. ABCD is a rhombus. AC and BD intersect at X. AD and AB are produced to E and F respectively, such that DE = BF. Show that:
   (a) $\angle AFC = \angle AEC$ (b) $\angle BFX = \angle DEX$
15. PQR is an equilateral triangle. Lines through R, Q and P parallel to PQ, PR and QR respectively, meet at X, Y, and Z, such that XPY, XQZ and YRZ are straight lines. Show that QY is perpendicular to XZ.

16. In a triangle ABC, AB = AC. H and K are points on AB and AC respectively, such that HK is parallel to BC. Show that ΔABK and ΔACH are congruent.

17. Two lines CB and ED meet at A when produced. A perpendicular line to AC through B meets a perpendicular line to AE through D at F. If DF = BF, show that $\angle$BAF = $\angle$DAF.

18. In a ΔXYZ, Q is a point on side YZ. The triangle is reflected in XZ to form the image XY′Z with Q′ the image of Q. Show that ΔQ′YZ is congruent to ΔQY′Z.

19. In figure 5.36, AB = DE, BC = DC and AC = CE. Show that AD = BE.

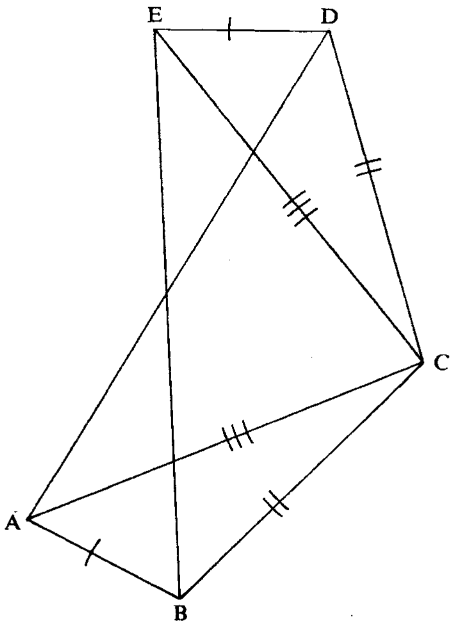


Fig. 5.36

Chapter Six

ROTATION

6.1: Introduction

In figure 6.1, triangle $A'B'C'$ is the image of triangle ABC under a transformation R.

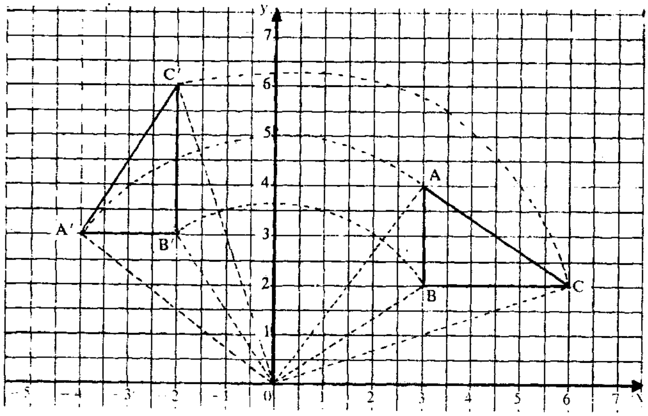


Fig. 6.1

To obtain A', the image of A, we rotate OA about O through an angle of 90^0 anticlockwise. Similarly, to obtain B', the image of B, we rotate OB about O through an angle 90^0 anticlockwise. The image of the point C is obtained in a similar way. Such a transformation is called a **rotation.** In this example, point O is called **the centre of rotation** and angle 90^0 is called the **angle of rotation.**

Note:

A rotation in the anticlockwise direction is taken to be positive, whereas a rotation in the clockwise direction is taken to be negative. For example, a rotation of 90^0 anticlockwise is $+90^0$ whereas the rotation of 90^0 clockwise is taken to be negative (-90^0)

In general, for a rotation to be completely defined, the centre and the angle of rotation **must** be stated.

Example 1

Figure 6.2 shows a triangle ABC in which AB = 3 cm. BC = 4 cm and AC = 5 cm. The triangle is rotated through $+60^\circ$ **about the point** X. Copy the figure and draw $\Delta\, A'B'C'$, the image of ΔABC.

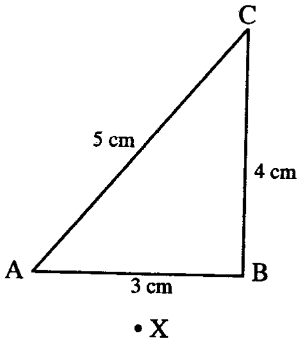


Fig. 6.2

Solution

To get $\Delta A'B'C'$, we proceed as follows:

(i) Draw angle AXL = 60^0 as in figure 6.3. To obtain A′ on XL, measure XA′ = XA.

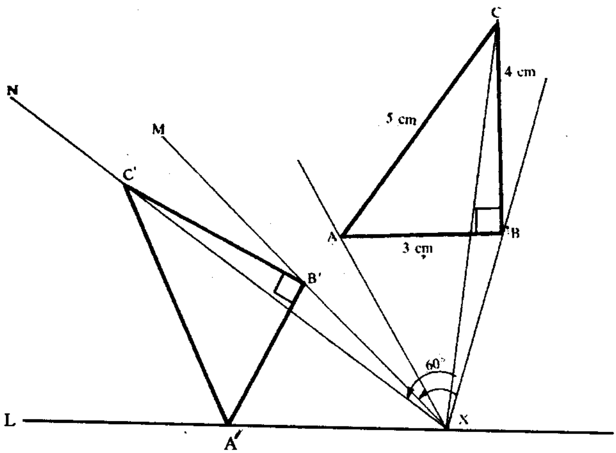


Fig. 6.3

(ii) Draw angle BXM = 60^0 as in the figure. To obtain B′ on XM, measure XB′ = XB.

(iii) Point C′ is similarly obtained by measuring XC′ = XC, see the figure.

(iv) Join A′B′. B′C′ and A′C′ to obtain $\Delta A'B'C'$. Alternatively, the image of ΔABC in figure 6.3 could be obtained by using tracing paper as follows:

(i) Make a copy of ΔABC and the point X in your exercise book. Join XA.

(ii) Mark off an angle of $+60^0$ at X.
(iii) Copy ΔABC and the line XA on tracing paper.
(iv) Place the tracing paper such that the drawing on it coincides with the one in the exercise book. Put a pin firmly at X and rotate the tracing through 60^0. You will know you have turned through 60^0 when XA on the tracing paper coincides with the other arm of the 60^0 angle on the exercise book.
(v) In the exercise book, mark the new position of ΔABC by pricking through its vertices. Join the points to obtain $\Delta A'B'C'$, the image of ΔABC.

With the same centre X, find the images of the ΔABC in figure 6.3 after a rotation of: (i) -60^0 (ii) $+45^0$ (iii) -120^0 (vi) 360^0

From figure 6.1 and 6.3, you should notice that:
(i) each point and its image are equidistant from the centre of rotation.
(ii) the object figure and its image are directly congruent.
(iii) the centre and the angle of rotation completely defines the rotation.

We can use the letter R to denote a rotation about a given point through a given angle. For example, if **R** represents a rotation, centre origin, through $+90^0$, then the image of P(5, –3) under **R** [R(P)] is (2,5).

6.2: Centre and Angle of Rotation

In figure 6.4, the square $A'B'C'D'$ is the image of the square ABCD after a rotation.

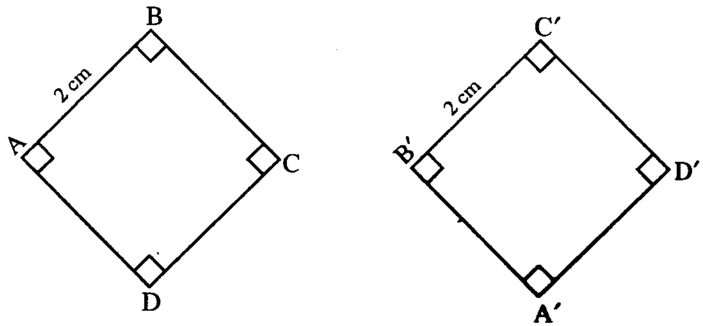


Fig. 6.4

To determine the centre and angle of rotation:
(a) Trace the figure. Join A to A' and construct a perpendicular bisector to AA'.
(b) Join D to D' and construct the perpendicular bisector to DD'.
The point of intersection of the two perpendicular bisectors is the centre

of rotation. Note that any two of the perpendicular bisectors to AA′, BB′, CC′ and DD′ are sufficient to locate the centre of rotation.

(c) Join O to A and A′. Measure ∠ AOA′. In this case, ∠ AOA′ = 90^0. This is the **angle of rotation.**

Note:

∠ AOA′ = ∠ BOB′ = ∠ COC′ = ∠ DOD′

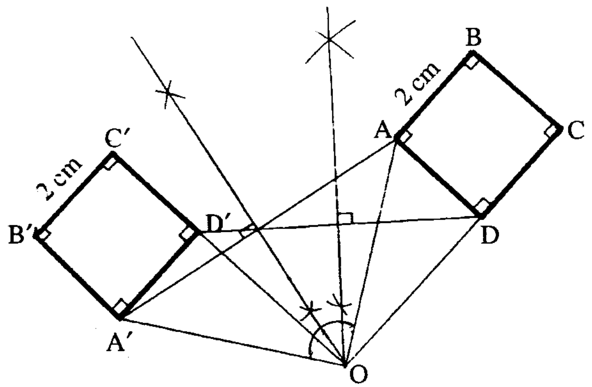
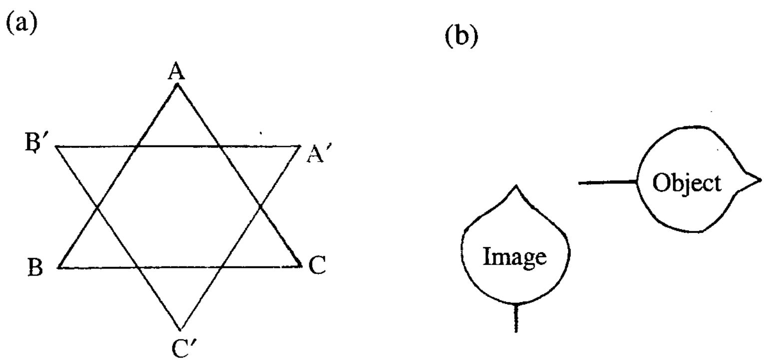


Fig. 6.5

A′B′C′D′ is the image of ABCD under a rotation through 90^0 in anticlockwise direction ($+90^0$).

Each pair of diagrams in figure 6.6 represents an object and its image after a rotation. Copy each and find the centre and angle of rotation:



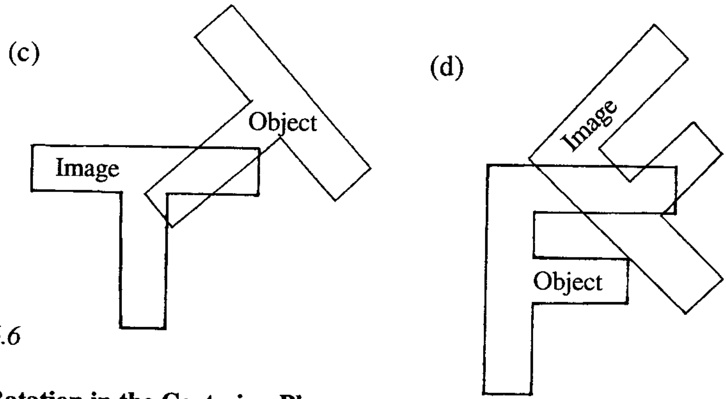


Fig. 6.6

6.3: Rotation in the Cartesian Plane

Rotation about the Origin through +90⁰, + 180⁰, + 360⁰

Figure 6.7 shows a point P and its images after rotations about (0, 0) with different angles of rotation:

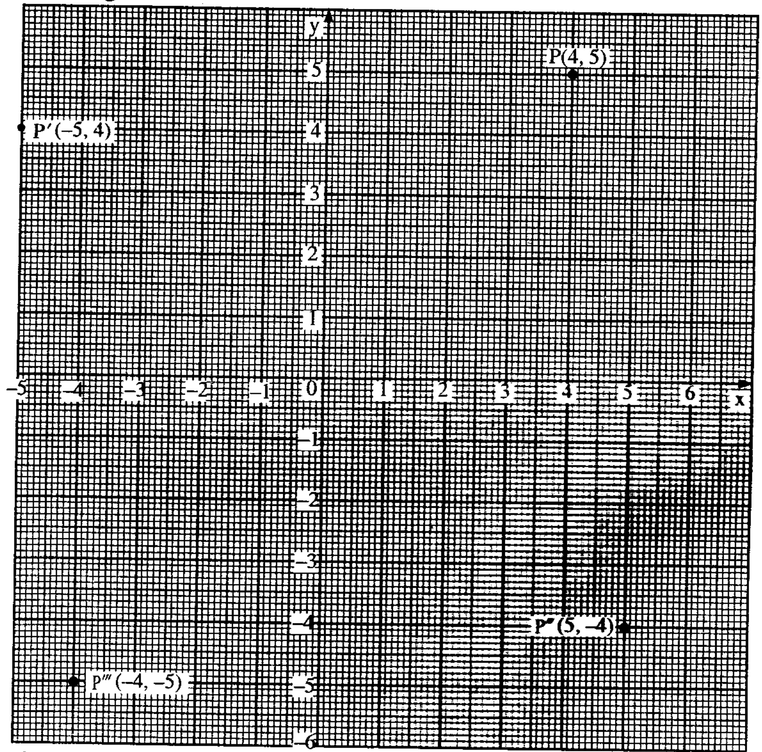


Fig 6.7

Table 6.1 below shows the images of P after a rotation about the origin through the indicated angles:

Table 6.1

Object point	P(4, 5)	P(4, 5)	P(4, 5)	P(4, 5)	P(4, 5)	P(4, 5)	P(4, 5)
Angle of rotation	$+90^0$	-90^0	$+180^0$	-180^0	$+360^0$	-360^0	0^0
Image point	$P'(-5, 4)$	$P''(5, -4)$	$P'''(-4, -5)$	$P(-4, -5)$	P(4, 5)	P(4, 5)	P(4, 5)

Note:

(i) A rotation through $\pm 180^0$ about (0, 0) gives the same image P'''

(ii) A rotation through $\pm 360^0$ or 0^0 about (0, 0) does not change the position of the object.

Repeat this for Q(3, 2), R(–1, 4) S(–2, 6), T(3, –1) and (a, b).

The following table shows the image of the point (a, b) through the indicated angle of rotation about the origin.

Table 6.2

Angle of rotation	0^0	$+90^0$	-90^0	$+180^0$	360^0	-360^0
Image of (a, b)	(a, b)	(–b, a)	(b, –a)	(–a, –b)	(a, b)	(a, b)

Rotation of a point (a, b) through 180^0, centre (x, y)

Consider the points (2, 3) and (7, 5). Taking (1, 4) as the centre, rotate the two points through 180^0.

Note:

(i) (2, 3) is mapped onto (0, 5). To obtain (0, 5) without a graph, consider the point (2, 3) and the centre of rotation (1, 4);
$(0, 5) = (2 \times 1 - 2 \times 4 - 3)$.

(ii) (7, 5) is mapped onto (–5, 3), which is obtained as follows:
$(-5, 3) = (2 \times 1 - 7), (2 \times 4 - 5)$.

Following the same pattern, the image of a point (a, b) rotated about (1, 4) through 180^0 is $(2 \times 1 - a, 2 \times 4 - b)$.

In general, a point (a, b) rotated through 180^0 about (x, y) is mapped onto the point (2x – a, 2y – b).

Write down the images of the following points under rotation through 180^0 about the stated centres of rotation:

(i) (3, –4) about (2, 1) (ii) (4, 5) about (3, 8)

(iii) (–5, 6) about (4, 4) (iv) (4, –7) about (–3, 2)

(v) (–2, –3) about (5, –3)
(vi) (–6, –8) about (–2, –5)
(vii) (7, 1) about (–3, –4)
(viii) (9, 2) about (–1, 3)
(ix) (–3, 6) about (7, 0)
(x) (0, 5) about (0, 2)

Exercise 6.1

1. A triangle whose vertices are $A'(-1.5, -2.5)$, $B'(-1.5, -1.5)$ and $C'(-3.5, -1.5)$ is an image of the triangle whose vertices are A(1.5, 2.5), B(1.5, 1.5) and C(3.5, 1.5) under a rotation. Find:
   (a) the centre and the angle of rotation.
   (b) the image of points (0, 3), (2, 2) and (0, 0) under the same rotation.
2. Describe the rotation which maps the square whose vertices are A(5, 3), B(6.5, 3), C(6.5, 1.5) and D(5, 1.5) onto the square whose vertices are $A'(8, -3)$, $B'(6.5, -3)$, $C'(6.5, -1.5)$ and $D'(8, -1.5)$. Find the image of the point (2, 0) under the same rotation.
3. Find the co-ordinates of the vertices of the image of a parallelogram whose vertices are A(3, 5), B(7, 5), C(5, 0) and D(1, 0) when rotated about the origin through:
   (a) -90^0.
   (b) 180^0.
4. A point P(–3, 2) maps onto $P'(1, -2)$ under a rotation R centre (1, 2). Find:
   (a) the angle of rotation.
   (b) $R(P')$.
5. The vertices of a trapezium are A(–3, 2), B(–2, 2), C(–1.5, 1.5) and D(–3.5, 1.5). The trapezium is rotated about (0.5, –4) through an angle of –60°. Find the co-ordinates of the vertices of its image. What is the image of (–1.5, 3.5) under the same rotation?
6. Give the co-ordinates of the image of each of the following points when rotated through 180^0 about (3, 1):
   (a) (–2, 0) (b) (4, 6) (c) (5, –3) (d) (–6, –6).
7. The transformation **R** represents a negative quarter turn about the point (–1, 0). The vertices of the image of a triangle ABC under **R** are $A'(3, 1)$, $B'(0, 5)$ and $C'(0, 1)$. Find the vertices of ΔABC.
8. The square whose vertices are O(0, 0), P(3, 0), Q(3, 3) and R(0, 3) is rotated through $+90^0$ to give an image whose vertices are $O'(3,-1)$, $P'(3,2)$, $Q'(0,2)$ and $R'(0,-1)$.
   (a) Find the centre of rotation.
   (b) If the square $O'P'Q'R'$ is reflected in the line x = 0, find the co-ordinates of O'', P'', Q'' and R''

9. Describe the rotation which maps the right-angled triangle with vertices at L(–4, –5), M(–2, –5) and N(–4, –3) onto the triangle with vertices at L′(–6, –3), M′(–6, –5) and N′(–4, –3). Find the image of the point (3, –4) under the same rotation

6.4: Rotational Symmetry of Plane Figures

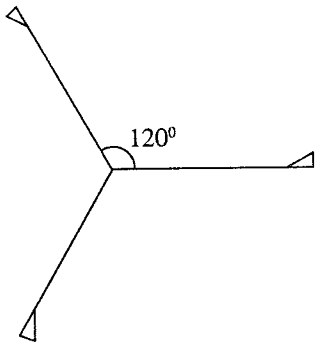


Fig. 6.8

Trace figure 6.8 above. Make a copy of tracing in your exercise book. Place the tracing on top of the copy and stick a pin through their centres so that thet racing can rotate. Rotate the tracing until it fits on top of the copy again. How many times can the figure fit onto itself in one complete turn?

The number of times the figure fits onto itself in one compete turn is called the **order of rotational symmetry.** Figure 6.8 has a rotational symmetry of order 3.

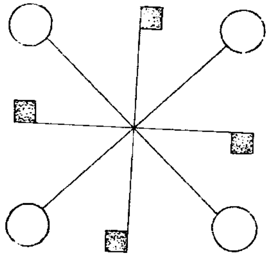
Note:

$$\text{The order of rotational symmetry of a figure} = \frac{360^0}{\text{angle between identical parts of the figure}}$$

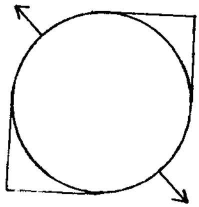
In figure 6.8, the angle between identical parts is 120^0. In plane figures, rotational symmetry is also referred to as **point symmetry.**

State the order of rotational symmetry of each of the following plane figures:

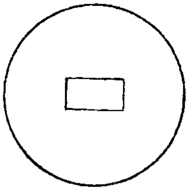
(i)



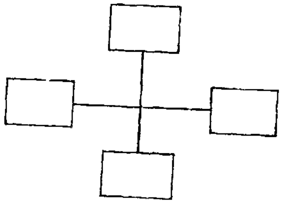
(ii)



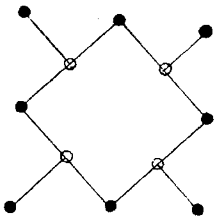
(iii)



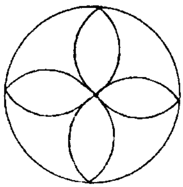
(iv)



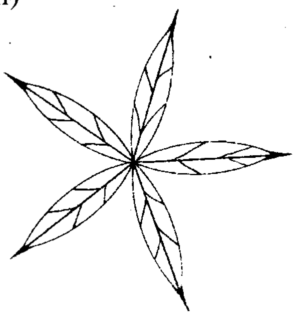
(v)



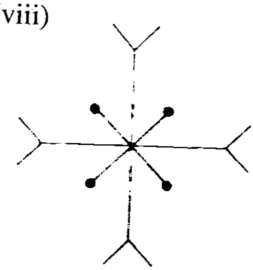
(vi)

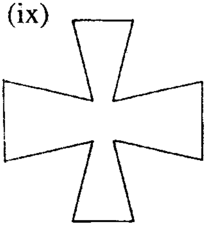


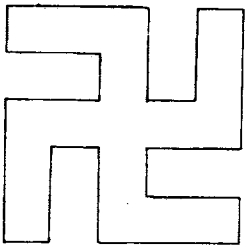
(vii)

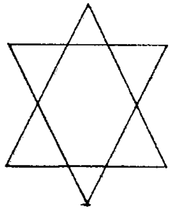


(viii)



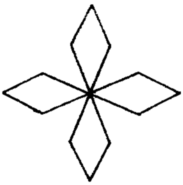
(ix)


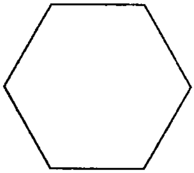
(x)


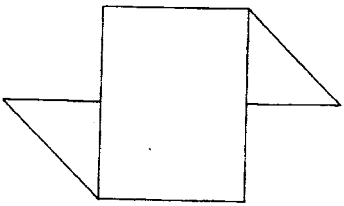
(xi)


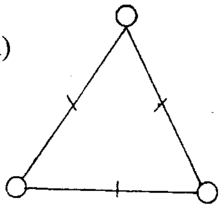
Exercise 6.2

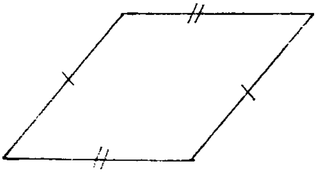
1. State the order of rotational symmetry of each of the following letters of the alphabet: A, B, H, I, N, O, S, X.
2. A figure has a rotational symmetry of order 2 about (–2, 2.5). Two of its vertices are (–2.5, 4) and (1.5, 4). Find the other vertices and draw the figure.
3. Trace and copy each of the following figures. Fill the order of rotational symmetry for each of the figures in table 6.3.

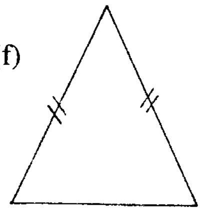
(a)


(b)


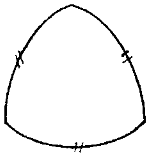
(c)


(d)


(e)


(f)


(g)



(h)

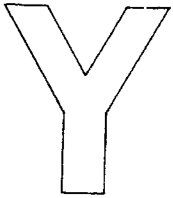


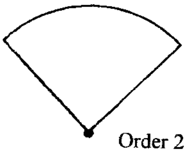
Fig. 6.10

Table 6.3

Diagram	a	b	c	d	e	f	g	h
Order of rotational symmetry								

4. Each of the following shapes is part of a symmetrical figure. For each, the point and order of rotational symmetry is indicated. Copy and complete the figures:

(a)

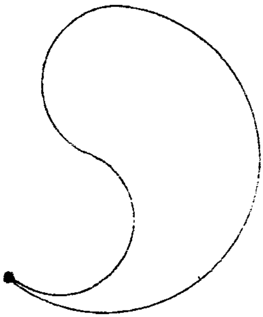


Order 2

(b)

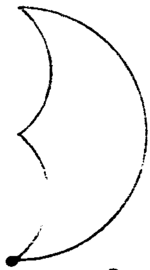
Order 4

(c)



Order 2

(d)

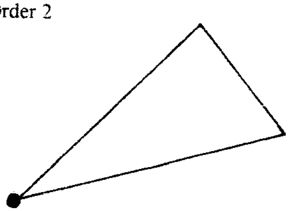


Order 6

(f)

Order 3

(e)



Order 5

Fig. 6.11

6.5: Rotational Symmetry of Solids

Figure 6.12 shows a model of pyramid with a square base. A straight stiff wire AB runs vertically through the vertex V and the centre C of the base:

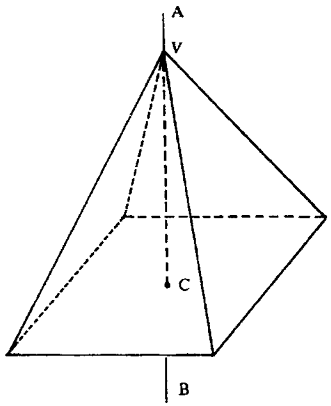


Fig. 6.12

If the pyramid is rotated about the stiff wire, it will fit onto itself four times in one complete turn. The pyramid therefore has a rotational symmetry of order 4 about its axis VC.

The line about which a solid has a rotational symmetry is called an **axis of symmetry.**

Other examples of Rotational Symmetry

A **triangular prism** whose cross-section is an equilaterial triangle has an axis of symmetry passing through the centre of triangular faces, as shown in figure 6.13: The prism has a rotational symmetry of order 3 about the axis.

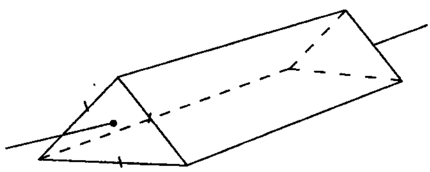


Fig. 6.13

The prism has also three other axes of symmetry each of order 2 such as the one shown in figure 6.14.

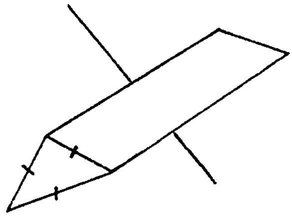


Fig. 6.14

A **cone** has one axis of rotational symmetry as shown in figure 6.14.

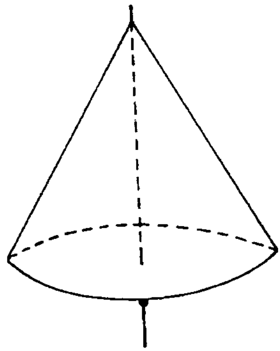


Fig. 6.15

The order of rotational symmetry is infinite. Why?

A **regular tetrahedron** has an axis of symmetry passing through one vertex and the centre of the opposite face as shown in figure 6.16 (a).

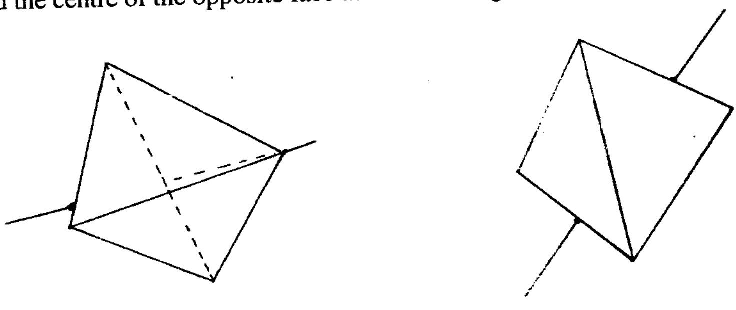


Fig. 6.16 *(a)* *(b)*

Fig. 6.16

(i) How many such axes does it have?

(ii) What is the order of rotational symmetry about each of these axes?

A regular tetrahedron has also an axis of symmetry passing through the midpoints of two opposite edges as shown in figure 6.17 (b).

(iii) How many such axes does it have?

(iv) What is the order of rotational symmetry about each of these axes?

(v) How many axes of symmetry has a regular tetrahedron?

6.6: Rotation and Congruence

In figure 6.18, triangle ABC is mapped onto triangle $A'B'C'$ after a rotation through –60° and centre X.

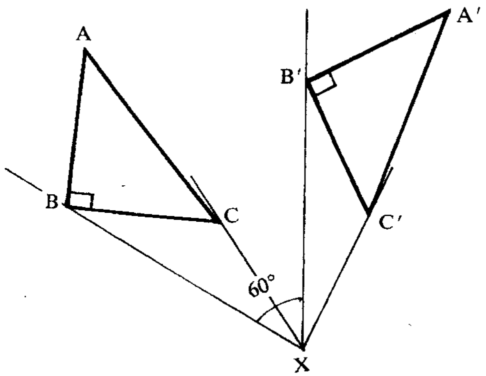


Fig. 6.17

Note:

The two triangles have:

(i) the same shape.

(ii) the same size.

The two triangles are therefore congruent. State the type of congruence.

In general, a rotation preserves lengths, angles and area, and the object and its image are **directly congruent.**

Exercise 6.4

1. Find:
   (i) the axes of symmetry,
   (ii) the order of rotational symmetry about each of the axes in the

following:

(a) Sunflower

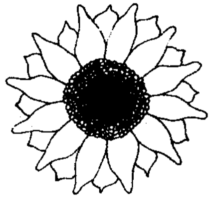


Fig. 6.18

(b) A water tower

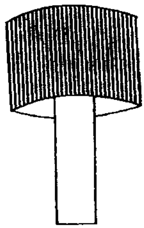


Fig. 6.19

(c) Milk packet

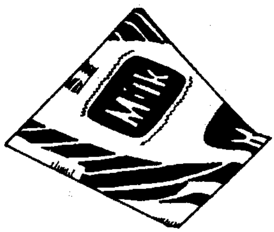


Fig. 6.20

(d) Queen cake

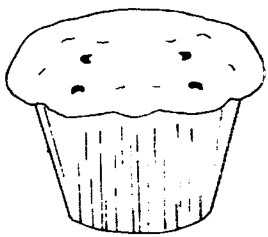


Fig. 6.21

(e) A hexagonal pencil

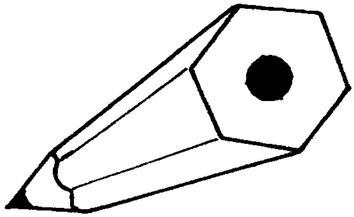


Fig. 6.22

(f) A car propeller

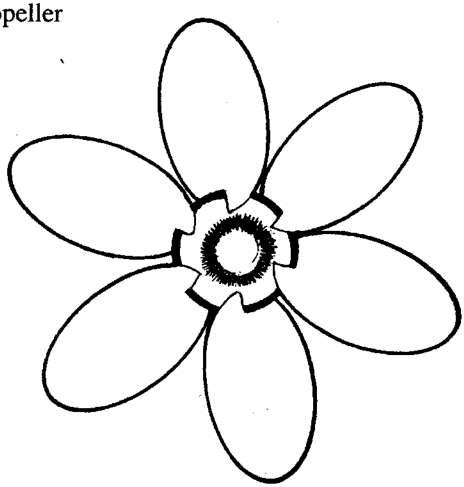


Fig. 6.23

2. Identify:
   (i) the axes of symmetry, and
   (ii) the order of rotational symmetry about each of the axes in each of the following solids:
   (a) cuboid.
   (b) cylinder.
   (c) hexagonal prism (pencil).
   (d) cube.
   (e) a regular octahedron.

Chapter Seven

SIMILARITY AND ENLARGEMENT

7.1: Similar Figures

Figure 7.1 shows two different sizes of a plan of a cross-section of a house.

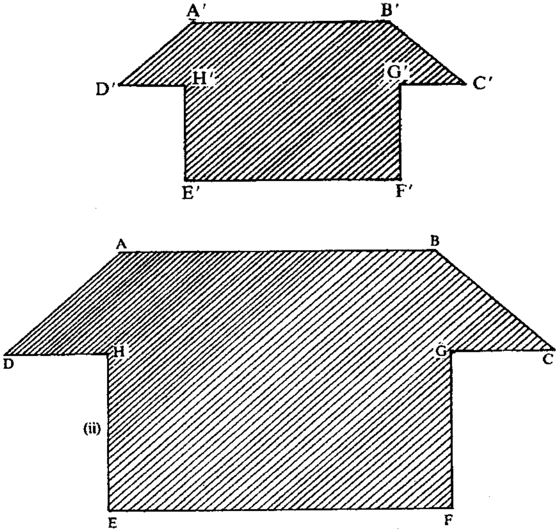


Fig. 7.1

Measure the corresponding sides of the two cross-sections. Copy and complete the following on the ratio of corresponding sides:

$\frac{AD}{A'D'} = 2$ $\qquad$ $\frac{\dots\dots}{E'F'} = 2$

$\frac{AB}{A'B'} = \dots\dots$ $\qquad$ $\frac{GF}{\dots\dots} = 2$

$\frac{DC}{\dots\dots} = 2$ $\qquad$ $\frac{DH}{D'H'} = \dots\dots$

Measure all the angles.

What do you notice about the corresponding **angles**?

You should notice that the ratio of **corresponding** sides is 2 and the corresponding angles are equal. The two **figures are** said to be **similar**.

In general, two or more figures are similar if:

(i) the ratio of the corresponding sides is constant.

(ii) the corresponding angles are equal.

Similar figures have the same shape, irrespective of size.

Example 1

In figure 7.2, the triangles PQR and WXY are similar. Calculate the lengths of PR and XY.

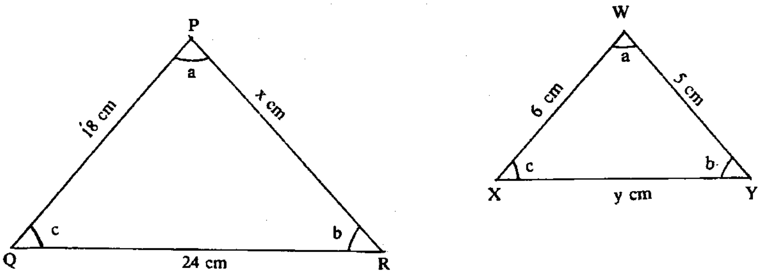


Fig. 7.2

Solution

PQ corresponds to WX (each of them is opposite angle b)

QR corresponds to XY and PR corresponds to WY.

Therefore, $\frac{PQ}{WX} = \frac{QR}{XY} = \frac{PR}{WY}$

That is, $\frac{18}{6} = \frac{24}{y} = \frac{x}{5}$

Thus, $\frac{18}{6} = \frac{24}{y}$

$18y = 24 \times 6$

$y = \frac{24 \times 6}{18}$

$y = 8$

Therefore, XY = 8 cm.

Also, $\frac{18}{6} = \frac{x}{5}$

$6x = 18 \times 5$

$x = \frac{18 \times 5}{6}$

$x = 15$

Therefore, PR = 15 cm

Note:
The corresponding sides of triangles are those opposite to equal angles.

Example 2
Figure 7.3 (a) and (b) shows two rectangles ABCD and EFGH whose measurements are as shown. Determine whether or not the two are similar.

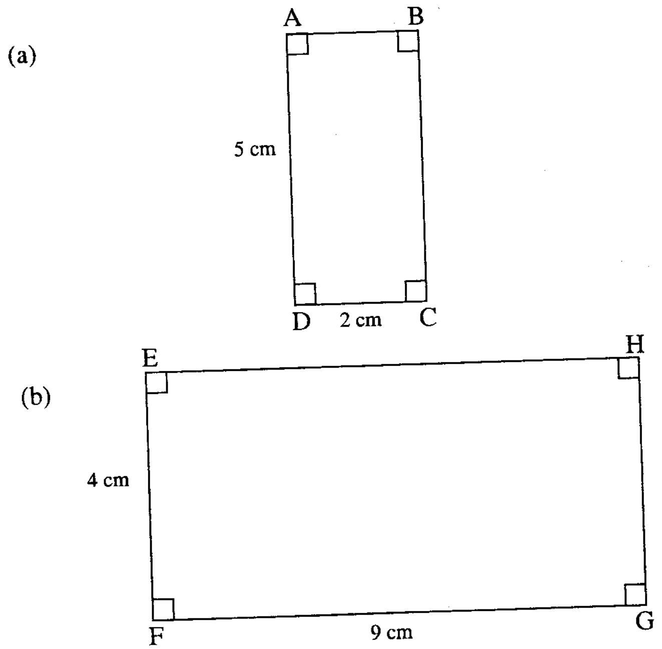


Fig. 7.3

Solution
The size of each angle is 90^0
The ratio of the corresponding sides are;

$$\frac{AD}{EH} = \frac{5}{9}$$

$$\frac{DC}{HG} = \frac{2}{4} = \frac{1}{2}$$

The ratio of their corresponding sides is not constant. Therefore, the two rectangles are not similar.

You should notice that although the corresponding angles of the two rectangles are equal, the rectangles are not similar because the ratios of their corresponding sides are not equal.

Note also that two figures could have the ratio of corresponding sides equal but fail to be similar if the corresponding angles are not equal. For example, a square of side 4 cm and a rhombus of side 5 cm are not similar.

(i) Construct two triangles with corresponding angles equal to 70°, 50° and 60°, with sides of different lengths. Find the ratio of the corresponding sides. Are the two triangles similar?

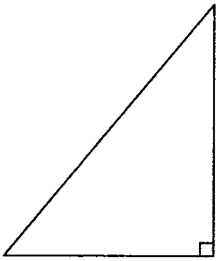
(ii) Construct two triangles whose side measure 2 cm, 3 cm, 4 cm, and 4 cm, 6 cm, 8 cm respectively. Measure the corresponding angles. Are the two triangles similar?

You should notice that **two triangles are similar if either all their corresponding angles are equal or the ratio of their corresponding sides is constant.**

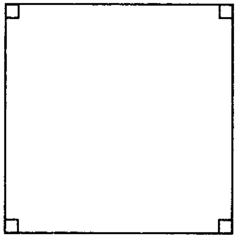
Exercise 7.1

1. Trace the following figures and determine those that are similar:

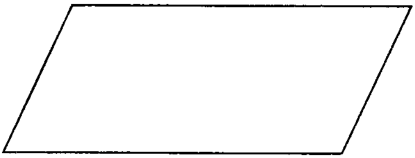
(a)



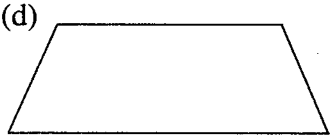
(b)



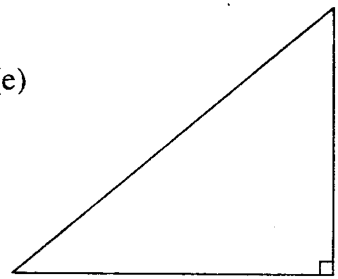
(c)



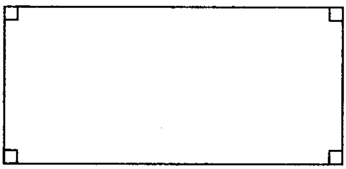
(d)



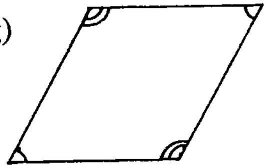
(e)



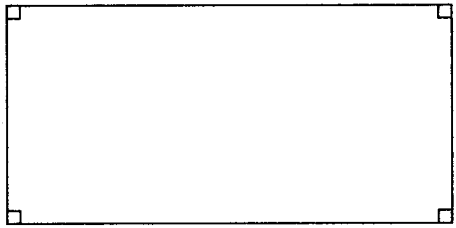
(f)



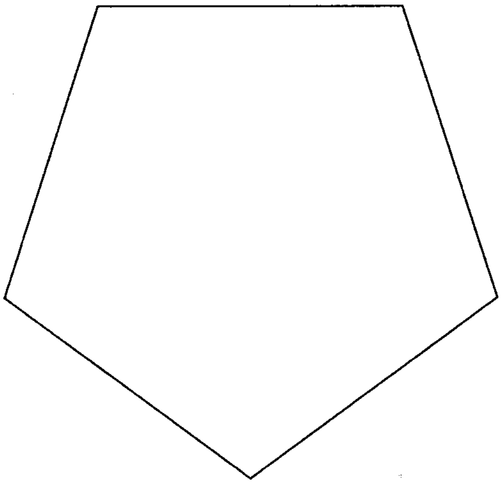
(g)



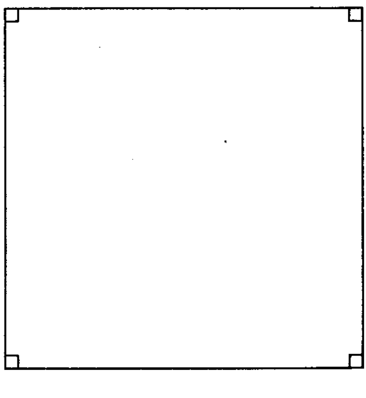
(h)



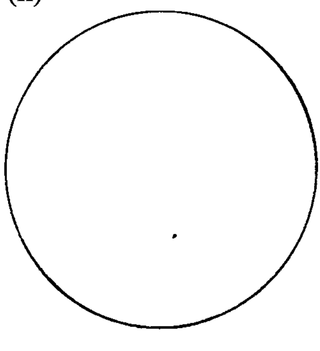
(i)



(j)



(k)



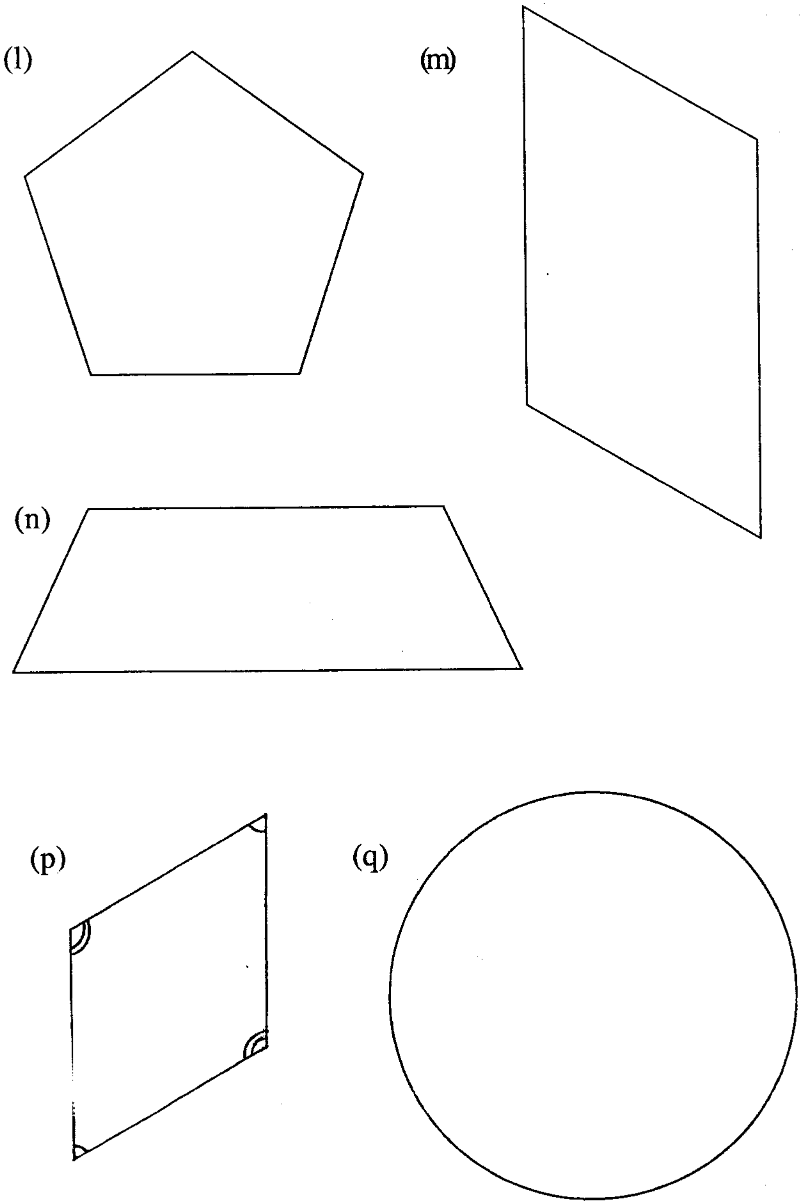


Fig. 7.4

In questions 2 to 4, identify the similar triangles

2. (a) (b) (c)

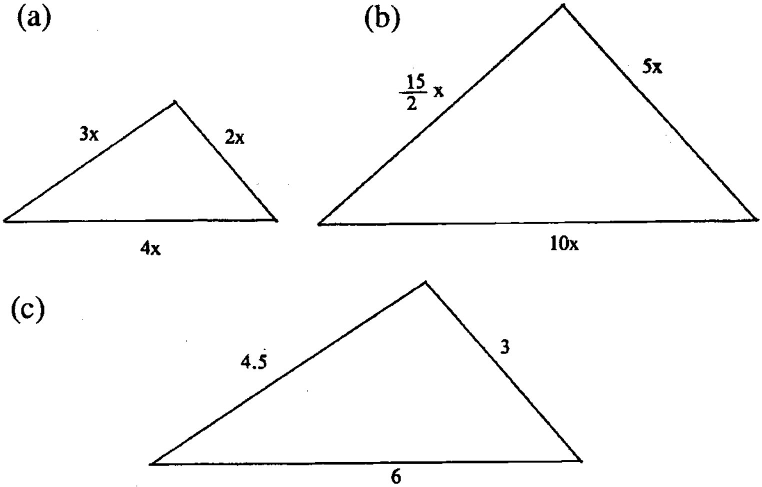


Fig. 7.5

3. (a) (b) (c)

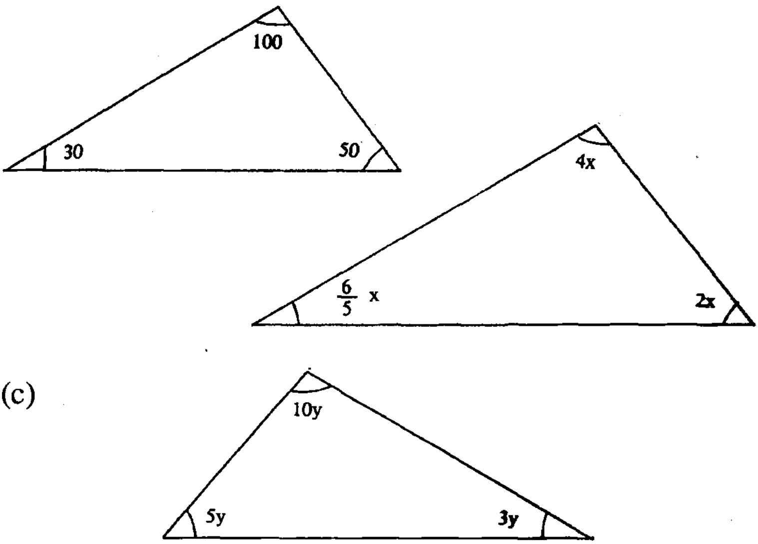


Fig. 7.6

4.

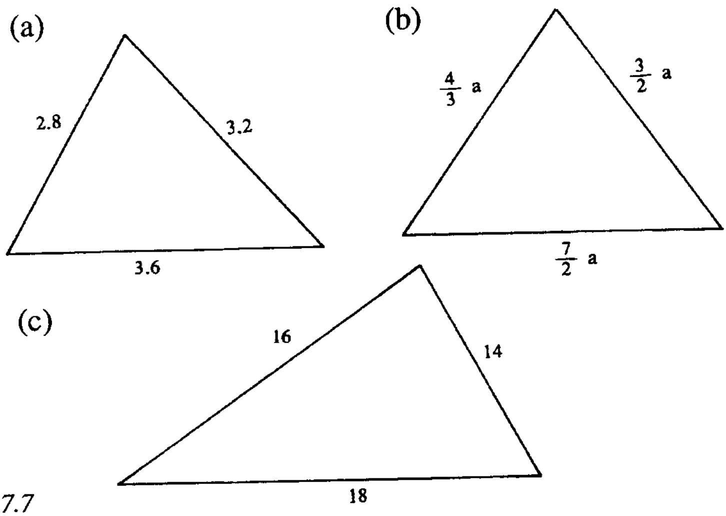


Fig. 7.7

5. In figure 7.8, identify all the geometric figures which are similar:

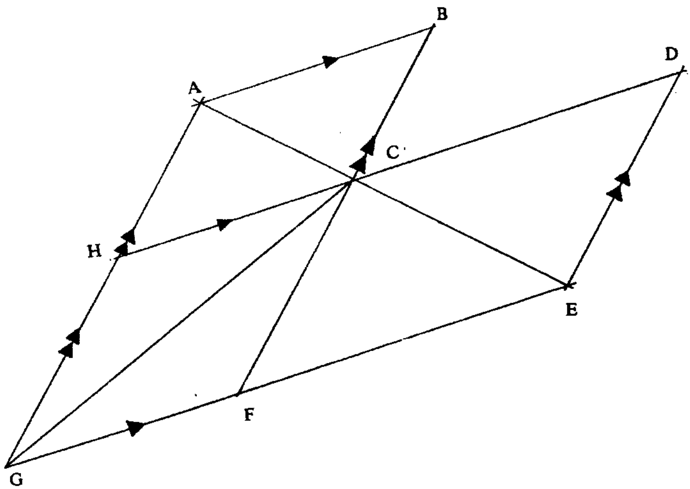


Fig. 7.8

6. In figure 7.9, name three pairs of:
   (a) corresponding sides.
   (b) corresponding angles.
   What can you say about triangles AOB and COD?

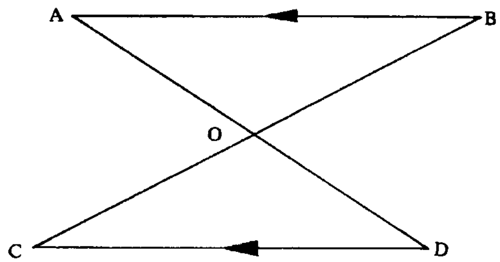


Fig. 7.9

7. In figure 7.10, triangle ABE is similar to triangle ACD. Calculate the length of BC.

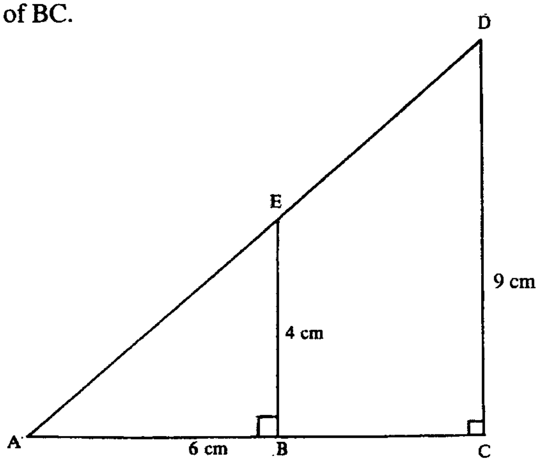


Fig. 7.10

8. In figure 7.11, SR is parallel to PQ. Calculate the lengths of SR and PX.

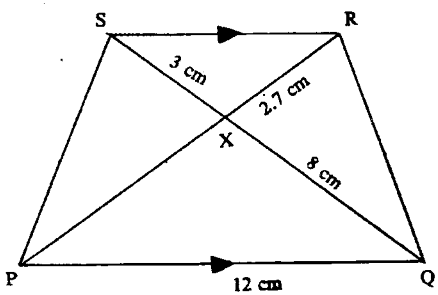


Fig. 7.11

9. In figure 7.12, $\angle EHG = \angle EFH = 90°$, HF = 5 cm and EF = 12 cm. Calculate the length HG and FG.

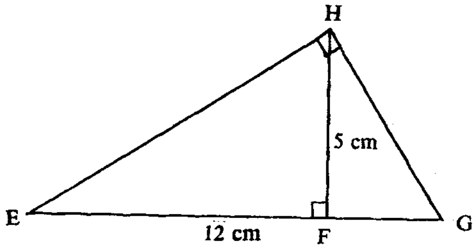


Fig 7.12

10. A triangle KLM is similar to a triangle PQR. KL = 5 cm, LM = 4 cm and $\angle KLM = 46°$. Construct triangle PQR such that QR = 6 cm.
11. ABCD and PQRS are similar rectangles. AB = 3 cm and BC = 7 cm. If PQ = 5 cm, construct rectangle PQRS.
12 WXYZ is a rectangle in which WX = 8 cm and XY = 6 cm. F is a point on YZ such that YF = 6 cm. Construct a rectangle YEFG similar to WXYZ such that YF is the diagonal.
13. In figure 7.13, $\angle TQS = \angle QPR$, PR//TS, PQ = 15 cm, QR = 6 cm and ST = 8 cm. Calculate QT.

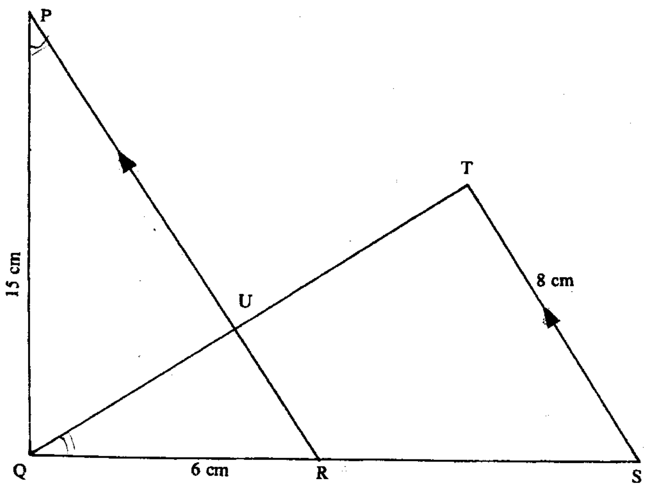


Fig. 7.13

7.2: Enlargement

In figure 7.14, PQR is a triangle in which PQ = 2 cm, QR = 1.5 cm and PR = 2.5 cm. O is a point such that OP = 4.1 cm, OQ = 2.5 cm and OR = 3.8 cm.

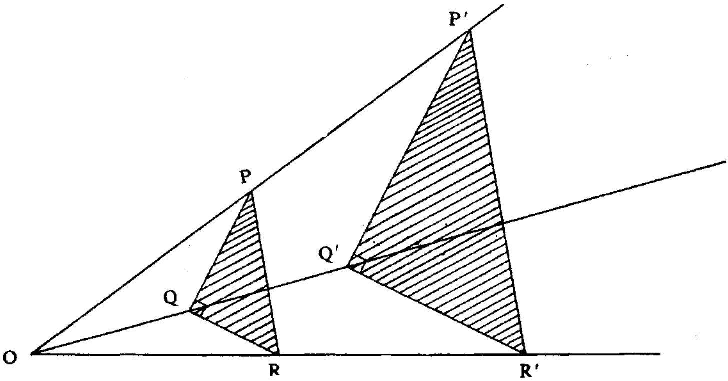


Fig. 7.14

Triangle $P'Q'R'$ has been obtained from ΔPQR as follows:
OP, OQ and OR are produced such that $OP' = 8.2$ cm, $OQ' = 5$ cm and $OR' = 7.6$ cm

This process of obtaining $\Delta P'Q'R'$ from ΔPQR is called **enlargement.** Triangle PQR is said to be the object and triangle $P'Q'R'$, its image under enlargement. The point O is called the centre of enlargement.

Note:

(i) $\dfrac{OP'}{OP} = \dfrac{OQ'}{OQ} = \dfrac{OR'}{OR} = 2$

(ii) $\dfrac{P'Q'}{PQ} = \dfrac{P'R'}{PR} = \dfrac{Q'R'}{QR} = 2$

This ratio is called the scale factor of enlargement. We realise that under enlargement, the object and its image are similar. The scale factor is referred to as the **linear scale factor.**

Example 3

Construct any triangle ABC. Take a point O outside the triangle. With O as the centre of enlargement and scale factor of 3, construct the image of ABC under the enlargement.

Solution

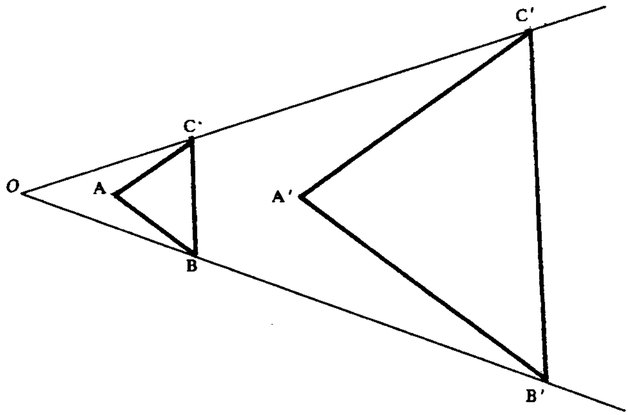


Fig. 7.15

By measurement;

OA = 1.5 cm, OB = 3 cm and OC = 2.9 cm. To get A', the image of A, we proceed as follows:

OA = 1.5 cm

$\frac{OA'}{OA} = 3$ (3 is the scale factor)

$OA' = 1.5 \text{ cm} \times 3$
$= 4.5 \text{ cm}$

Produce OA, and measure 4.5 cm from O to get A'.

Similarly;

$\frac{OB'}{OB} = 3$

$OB' = 3 \times 3 = 9 \text{ cm}$

$\frac{OC'}{OC} = 3$

$OC' = 2.9 \times 3$
$= 8.7 \text{ cm}$

Produce OB and OC to obtain B' and C' respectively.

Note:

Lines joining object points to their corresponding image points meet at the centre of enlargement. We use this fact to locate the centre of enlargement if we are given an object and its image.

Example 4

Triangle A′B′C′ is the image of triangle ABC under an enlargement.

(a) Locate the centre of the enlargement.

(b) Find the scale factor of the enlargement.

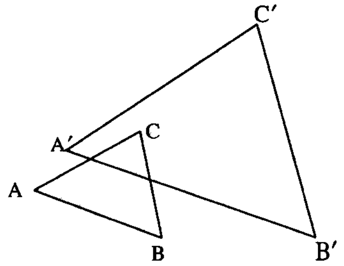


Fig. 7.16

Solution

(a) To locate the centre of enlargement, proceed as follows:
Join A to A′, B to B′ and C to C′ and produce them. The point O where the lines meet is the centre of enlargement.

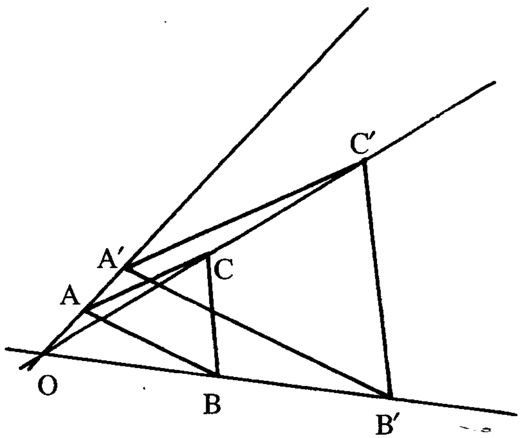


Fig. 7.17

(b) By measurement;

$$\frac{OA'}{OA} = 2$$

$$\frac{OB'}{OB} = 2$$

$$\frac{OC'}{OC} = 2$$

Therefore, the scale factor of the enlargement is 2.

Example 5

In figure 7.18, $\Delta P'Q'R'$ is the image ΔPQR under an enlargement, centre O.

(a) If $OQ = 6$ cm and $QQ' = 4$ cm, find the scale factor of the enlargement.

(b) If $PQ = 4$ cm, calculate the length of $P'Q'$.

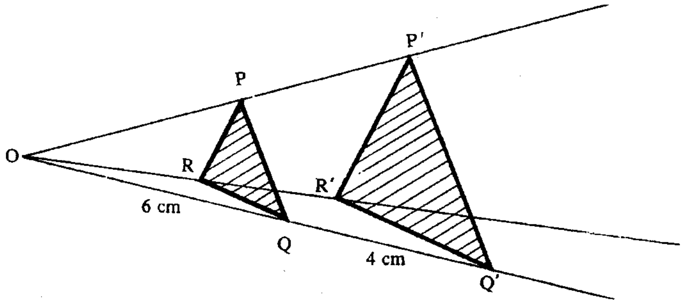


Fig. 7.18

Solution

(a) Linear scale factor is $\frac{OQ'}{OQ} = \frac{(6+4)}{6} = \frac{10}{6} = \frac{5}{3}$

(b) Linear scale factor = $\frac{P'Q'}{PQ}$

But $PQ = 4$ cm

Therefore, $\frac{P'Q'}{4} = \frac{5}{3}$

$P'Q' = 6\frac{2}{3}$ cm

Example 6

In figure 7.19, rectangle $A'B'C'D'$ is the image of rectangle ABCD under an enlargement with centre at O. $OA = 12$ cm, $OA' = 4$ cm, $AB = 6$ cm and $A'D' = 3$ cm.

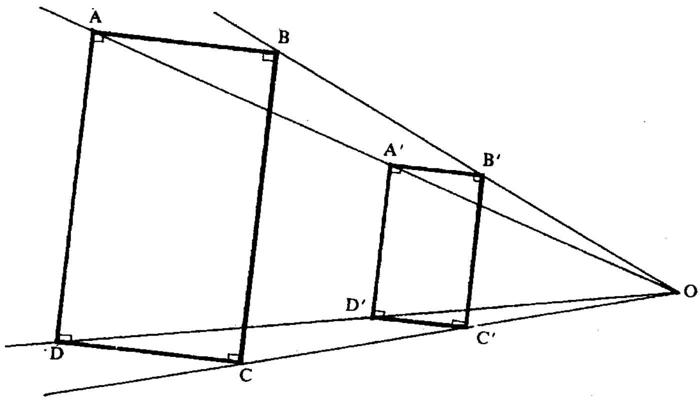


Fig. 7.19

Calculate:

(a) the linear scale factor.

(b) the length of $A'B'$.

(c) the length of BC.

Solution

(a) The linear scale factor is given by;

$$\frac{OA'}{OA} = \frac{4}{12} = \frac{1}{3}$$

(b) $\frac{A'B'}{AB} = \frac{1}{3}$

Therefore, $A'B' = 6 \times \frac{1}{3} = 2$ cm

(c) $\frac{B'C'}{BC} = \frac{1}{3}$

Therefore, $BC = 3\,B'C'$
$= 3 \times 3$
$= 9$ cm

Note that when the scale factor is a proper fraction, as in this case, the image is smaller than the object.

Figure 7.20 shows a rhombus of side 3 cm. Trace the figure. Using O as the centre of enlargement and linear scale factor 2, draw the image of the rhombus.

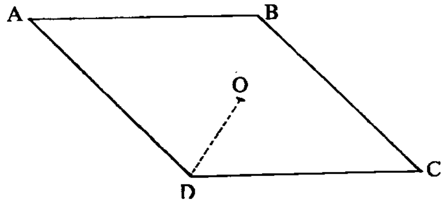


Fig. 7.20

Exercise 7.2

In questions 1 to 4, copy the figures using a tracing paper

1. In figure 7.21, locate the image of AB under enlargement with centre F and scale factor of:

   (a) 2 (b) 3 (c) $\frac{1}{2}$

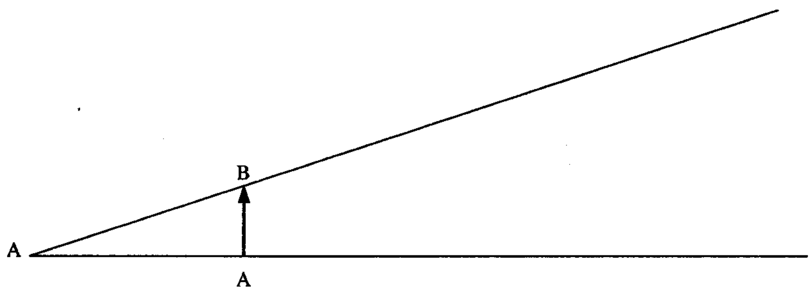


Fig. 7.21

2. Draw the image of the square ABCD in figure 7.22 under an enlargement scale factor 3 and centre X.

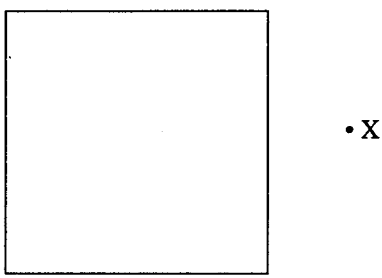


Fig. 7.22

3. Find the image of (a), (b) and (c) in figure 7.22, given the centres of enlargement marked X and the factor 1.5:

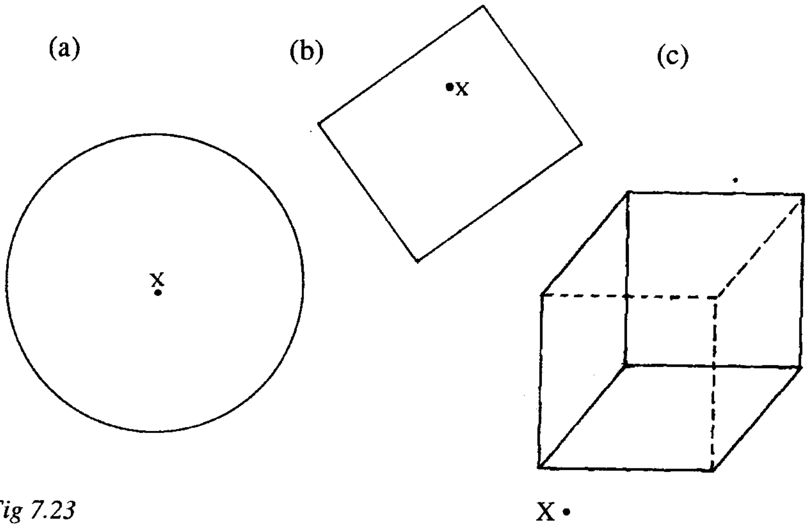


Fig 7.23

4. Using B as centre of enlargement and a scale factor of $\frac{1}{2}$, find the images of (a) and (b) in figure 7.24

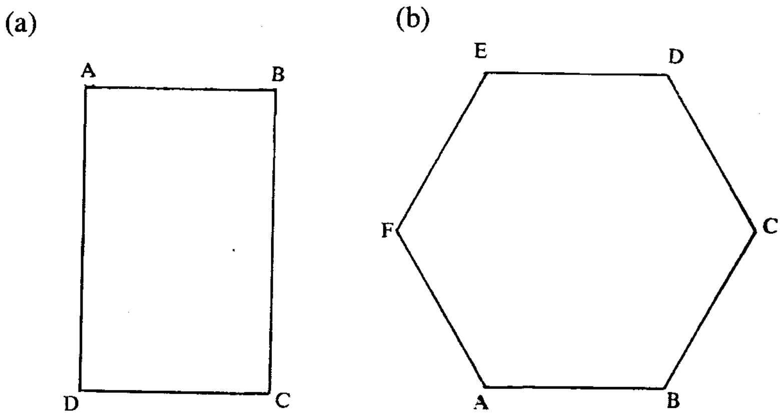


Fig. 7.24

5. The vertices of an object and its image after an enlargement are A(0, 0), B(1, 1), C(2, 0), D(1, –1) and A′(3, 3), B′(5, 5), C′(7, 3), D′(5, 1) respectively. Find the centre and scale factor of the enlargement.
6. Given that P(3, 4), Q(4, 4), R(6, 4), S(7, 1) and T(5, 0) are the vertices of an object, find the vertices of the image after an enlargement with the centre at (0, 0) and scale factor:

(a) $\frac{1}{2}$

(b) 3

Negative Scale Factor

In figure 7.25, P′Q′R′ and P″Q″R″ are the images of PQR under an enlargement, centre O. Both images are twice as large as PQR:

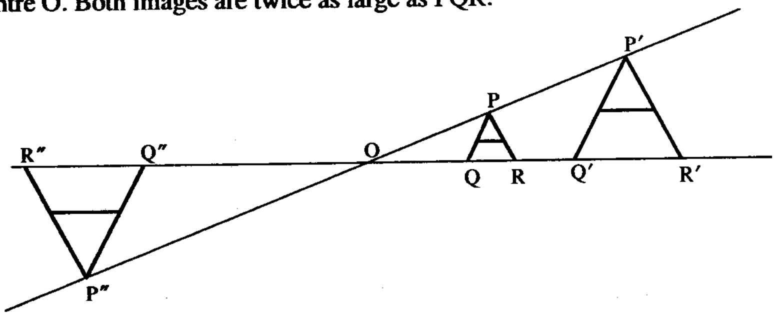


Fig. 7. 25

P′Q′R′ is the image of PQR under an enlargement centre 0, scale factor 2. How can we describe P″Q″R″, the image of PQR under an enlargement with 0 as the centre?

Note:

(i) PQR and its image P″Q″R″ are on opposite sides of the centre of enlargement.

(ii) the image is inverted.

When this happens, the scale factor is said to be negative. In this case, the scale factor is –2

Note:

To locate the image of the object under an enlargement with a negative scale factor, the same procedure as for the positive scale factor is followed. However, the object and the image fall on opposite sides of the centre of enlargement.

Exercise 7.3

1. Points A(4, 2), B(9, 2), C(7, –2) and D(2, –2) are the vertices of a parallelogram. Taking the origin as the centre of enlargement, find the image when the scale factor is:

   (a) –1 (b) –2 (c) $-\frac{1}{4}$

2. State the centre of enlargement and the linear scale factor in each of the following figures if $\Delta A'B'C'$ is the image of ΔABC:

(a)

(b)

(c)

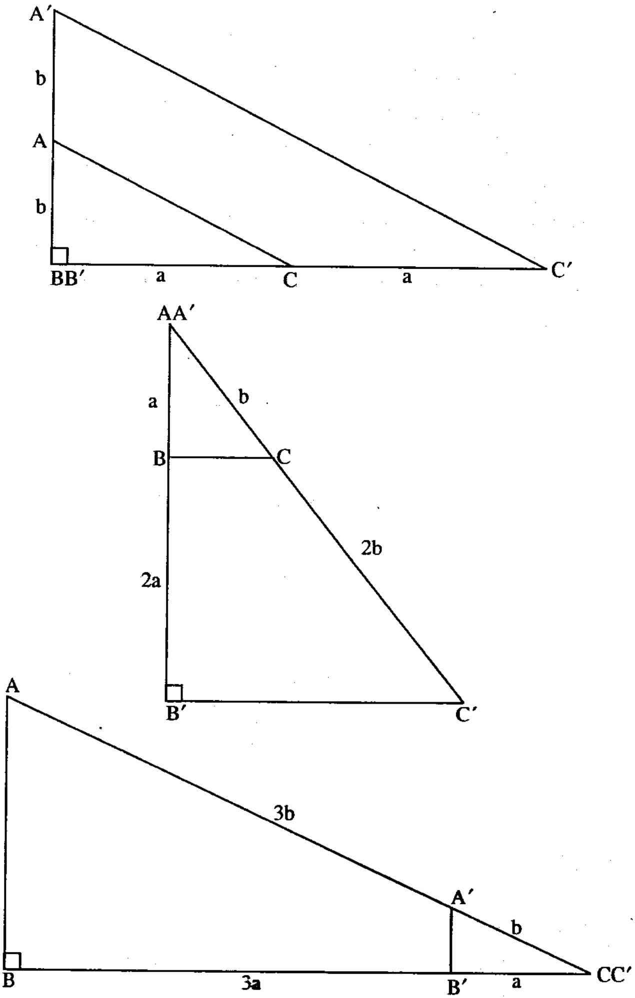


Fig. 7.26

3. In a ΔABC, $AB = 3$ cm, $BC = 4$ cm and $AC = 5$ cm. The sides AB, BC and AC are enlarged by a linear scale factor of 2.5. What are the corresponding lengths of the new triangle?

4. If a road 8 km long is represented by 4 cm on a map, what is the scale of this map?

5. A rectangle measures 3 cm by 5 cm. Find the corresponding measurements of the image of the rectangle after an enlargement scale factor:
   (a) -0.75 (b) -1.5 (c) -3

6. The radius of a circle is 3.5 cm. Find the circumference of its image after an enlargement with scale factor 1.75.

7. A triangular field PQR is such that $PQ = 20$ m, $PR = 30$ m and $QR = 40$ m. The field is drawn to scale such that the side PQ is 3 cm long. Find the linear scale factor and the lengths of OR and PR.

8. Figure 7.27 shows a cone of base radius 28 cm and the slant side of length 35 cm. At a point P, 14 cm vertically below the vertex, the cone is cut across to form a smaller one. Calculate the base radius of the smaller cone.

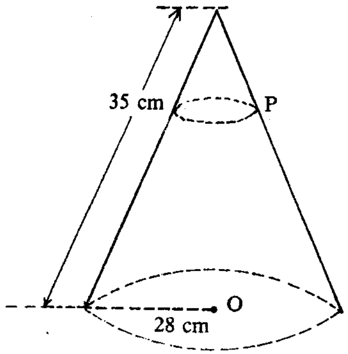


Fig. 7.27

7.3: Area Scale Factor

In figure 7.28, $A'B'C'D'$ is the image of rectangle ABCD after an enlargement whose scale factor is 2.

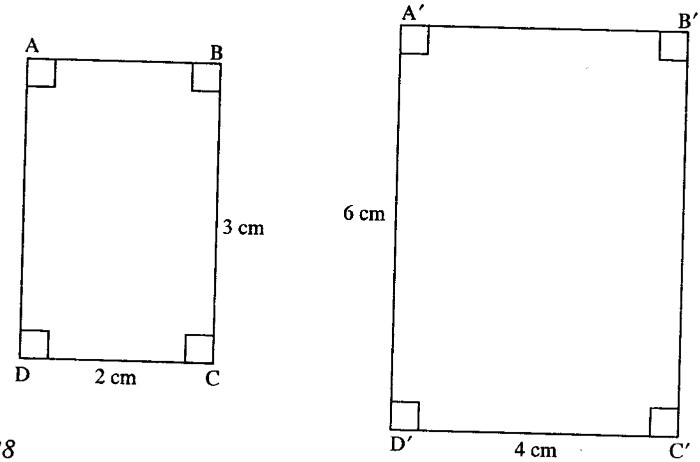


Fig. 7.28

Area of ABCD is $(2 \times 3)\ \text{cm}^2 = 6\ \text{cm}^2$

Area of A′B′C′D′ is $(4 \times 6)\ \text{cm}^2 = 24\ \text{cm}^2$

$$\frac{\text{Area of } A'B'C'D'}{\text{Area of } ABCD} = \frac{24\ \text{cm}^2}{6\ \text{cm}^2} = 4$$

The ratio of area of image to area of object is called **area scale factor.**

A triangle of base 3 cm and height 4 cm is given an enlargement with scale factors:

(a) 3 (b) $\frac{1}{2}$ (c) -2

In each case, find:

(i) the corresponding dimensions of the three images and hence their areas.

(ii) the area scale factors.

You should notice that **area scale factor is the square of linear scale factor.**

Example 7

A triangle whose area is 12 cm^2 is given an enlargement with linear scale factor 3. Find the area of the image.

Solution

Linear scale factor is 3.

Area scale factor is $3^2 = 9$

$$\frac{\text{Area of image}}{12} = 9$$

Therefore, area of image is 12 x 9 = 108 cm^2

Example 8

A map of a certain town is drawn to a scale of 1: 50 000. On the map, the railway quarters cover an area of 10 cm^2. Find the area of the railway quarters in hectares.

Solution

The linear scale factor is $\frac{50\,000}{1} = 50\,000$

The area scale factor is $50\,000^2$. Therefore, 10 cm^2 on the map would represent $50\,000^2 \times 10$ cm^2 on the ground. In hectares, this would be;

$$\frac{50\,000 \times 50\,000 \times 10}{100 \times 100 \times 10\,000} = 250 \text{ ha}$$

Example 9

The ratio of the area of two circles is $\frac{16}{9}$.

(a) What is the ratio of their radii?

(b) If the larger one has a radius of 20 cm, find the radius of the smaller one.

Solution

(a) Area scale factor = $\frac{16}{9}$

Therefore, linear scale factor is $\sqrt{\left(\frac{16}{9}\right)} = \frac{4}{3}$

Hence, the ratio of the radius of the smaller to the radius of the larger is 3 : 4.

(b) If the radius of the larger circle is 20 cm, then the radius of the smaller circle is; $20 \div \frac{4}{3} = 20 \times \frac{3}{4}$

$= 15$ cm

Exercise 7.4

1. A plan of a house measures 20 cm by 13 cm. Find the area of the actual house if the linear scale factor is 50.
2. The corresponding lengths of two similar photographs are 12 cm and 30 cm. The area of the larger photograph is 750 cm^2. Find:
   (a) the area scale factor.
   (b) the area of the smaller photograph.
3. The floor of a room measuring 15 m by 9 m requires 3 375 square tiles. What is the measurement of each tile? If the tiles available were twice as long, how many would be required?

4. The ratio of the area of two similar rooms is $\frac{4}{25}$.
   (a) Find the area of the bigger room if the area of the smaller room is 8 m^2.
   (b) Find the ratio of their lengths.
   (c) If the length of the larger room is 10 m, find the length of the smaller one.
5. The corresponding sides of two similar regular pentagons are 3 cm and 7 cm respectively.
   (a) Find the ratio of their areas.
   (b) Calculate the area of the larger if the area of the smaller is 36 cm^2.
6. Mwangi has two balloons, one yellow and the other green. The radius of the yellow balloon is 7 cm while that of the green balloon is 21 cm. Find the ratio of their surface areas.

7.4: Volume Scale Factor

Figure 7.29 represents two similar cuboids, the big one measuring $2\text{ cm} \times 3\text{ cm} \times 5\text{ cm}$ and the smaller one $1\text{ cm} \times 1.5\text{ cm} \times 2.5\text{ cm}$. The linear scale factor is 2.

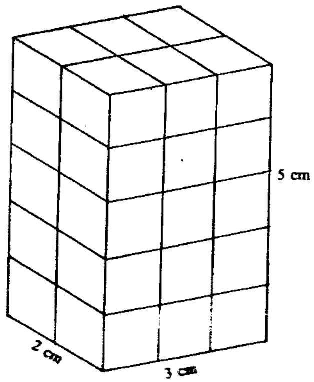
The ratio of the volume of the large cuboid to the volume of the smaller cuboid is;

$$\frac{(2 \times 3 \times 5)\text{ cm}^3}{(1 \times 1.5 \times 2.5)\text{ cm}^3} = 8$$

$$= 2^3 \text{ (2 is the linear scale factor).}$$

This ratio is called **volume scale factor.**

(a)



(b)

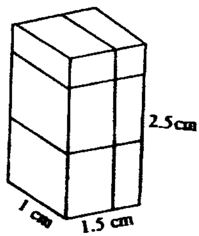


Fig. 7.29

Note:
When the linear scale factor is 2, the volume scale factor is 2^3.
Volume scale factor = (linear scale factor)^3

Example 10
The base radii of two similar cones are 6 cm and 8 cm. If the volume of the smaller cone is 324 cm^3, find the volume of the larger one.

Solution
The ratio of their radii is $\frac{8 \text{ cm}}{6 \text{ cm}} = \frac{4}{3}$

Linear scale factor is $\frac{4}{3}$.

Therefore, the volume scale factor is $\left(\frac{4}{3}\right)^3 = \frac{64}{27}$

Thus, volume of the larger cone $\frac{64}{27} \times 324 = 768 \text{ cm}^3$

Example 11
A model of a swimming pool is shown in figure 7.30. Find the capacity in litres of the actual swimming pool if its length is 50 metres.

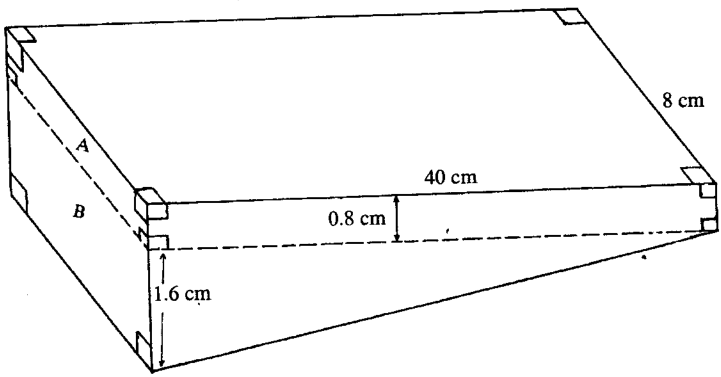


Fig. 7.30

Volume of A $= 40 \times 8 \times 0.8$
$= 256 \text{ cm}^3$

Volume of B $= \frac{1}{2} \times 1.6 \times 40 \times 8$
$= 256 \text{ cm}^3$

Total volume $= 512 \text{ cm}^3$

$$\text{Linear scale factor} = \frac{\text{actual length of pool in cm}}{\text{length of the model}}$$

$$= \frac{50 \times 100}{40}$$

$$= 125$$

Volume scale factor is given by;
$125^3 = 1\,953\,125$

Actual capacity of the pool; $\frac{1\,953\,125 \times 512}{1\,000} = 1\,000\,000$ litres

Exercise 7.5

1. The volume scale factor of two similar cylinders is 27. Find :
   (a) the linear scale factor.
   (b) the area scale factor.
2. The corresponding sides of two similar blocks of wood measure 15 cm and 30 cm. Find the ratio of their volumes.
3. A spherical balloon 5 cm in a diameter is inflated so that its diameter is 10 cm. By what factor has the volume increased?
4. The corresponding lengths of two similar iron bars are 5 cm and 15 cm.
   (a) What is the ratio of their masses?
   (b) If the smaller iron bar has a mass of 12 kg, what is the mass of the larger bar?
5. A spherical solid lead of diameter 12.0 cm weighs 6.4 kg. How much would a similar solid of a diameter 10.0 cm weigh?
6. The heights of two similar pails are 12 cm and 8 cm. The larger pail can hold 2 litres. What is the capacity of the smaller pail?
7. The mass of fat in two similar tins are 2 000 g and 500 g respectively. If the area of the base of the smaller tin is 100 cm^2, find the area of the base of the larger tin.
8. A soft drink manufacturing company makes a giant model bottle 2 m long as the factory symbol. If the real bottle from the factory is 20 cm long and has a volume of 300 cm^3, find the volume of the model in cm^3.
9. The shortest side of a triangle is 12 cm and the area of the triangle is 8 cm^2. A similar triangle has an area of 18 cm^2. Calculate the shortest side of this triangle.
10. The volumes of two similar cans are 96 cm^3 and 1 500 cm^3. Find the ratio of:
    (a) their heights.
    (b) the area of their curved surfaces.

11. The radius of a soap bubble increases by 4%. Calculate the percentage increase in its:
    (i) surface area.
    (ii) volume (to three significant figures).
12. The dimensions of a triangle PQR are 16 cm, 20 cm and 28 cm. The dimensions of a similar triangle LMN are 24 cm, 30 cm, and 42 cm respectively. Calculate the ratio of:
    (a) their perimeters.
    (b) their areas.
13. In figure 7.31, VAB, VCD and VEF are similar cones. Their radii are 14 cm, 10.5 cm and 7 cm respectively. If the height of VAB is 42 cm, calculate:
    (a) the heights of the cones VCD and VEF.
    (b) the ratio of the curved surface areas of cones VCD and VEF.
    (c) the ratio of their volumes.

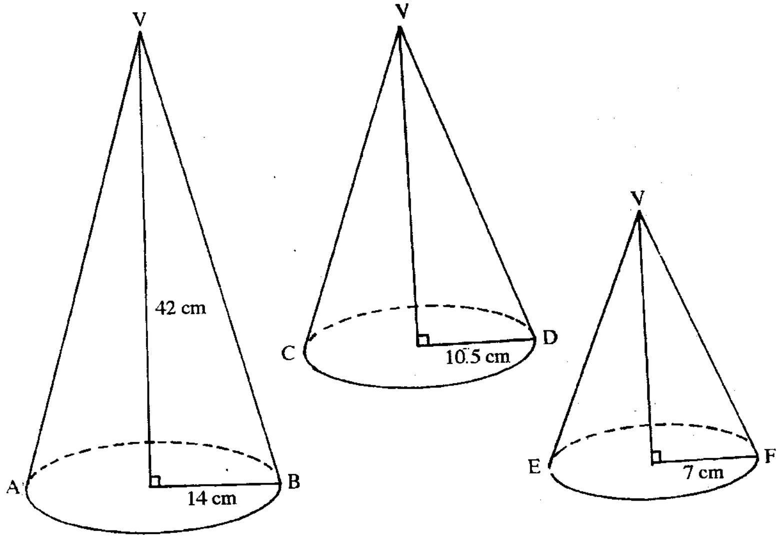
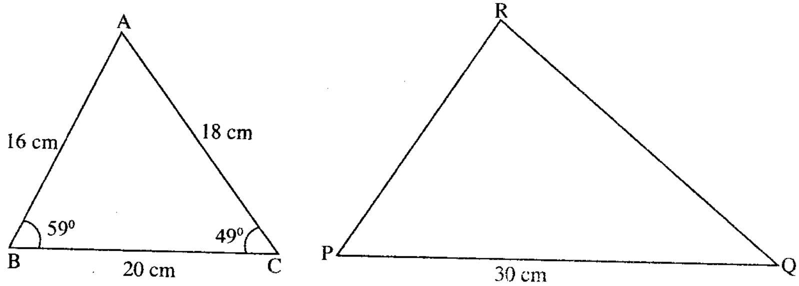


Fig. 7.31

Mixed Exercise 1

1. Evaluate: $\sqrt{\left(\dfrac{0.64 \times (1.69)^{\frac{1}{2}}}{(0.04)^{\frac{1}{2}} \times 38.44}\right)}$

2. Given that the triangles ABC and PQR in the figure below are similar, find:
   (a) the size of $\angle QPR$.
   (b) the lengths of RQ and PR.



3. Evaluate $\dfrac{a^b - b^a}{b^b \times a^a}$, if a = 2 and b = –2.

4. Find y if $\log_2 y - 2 = \log_2 92$.

5. Given that $y = ax^n$, find the value of y when $a = \frac{5}{3}$, x = 2 and n = –4.

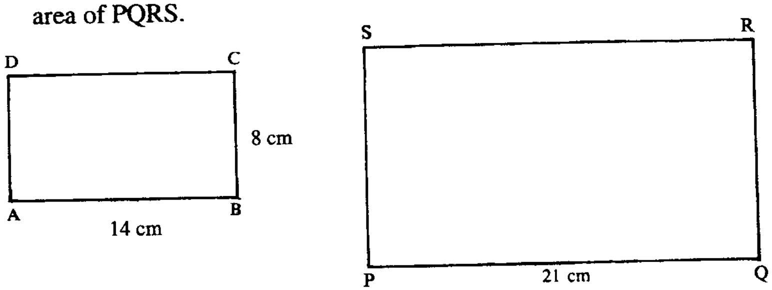
6. A straight line l_1 has a gradient $-\frac{1}{2}$ and passes through the point P(–1, 3). Another straight line l_2 passes through the points Q(1, –3) and R(4, 5). Find:
   (a) the equation of l_1.
   (b) the gradient of l_2.
   (c) the equation of l_2.
   (d) the co-ordinates of the point of intersection of l_1 and l_2.
   (e) the equation of a line through R parallel to l_1.
   (f) the equation of a line passing through a point S(0, 5) and perpendicular to l_2.

7. Given that $\log y = 3.143$ and $\log x = 2.421$, evaluate:

   (a) $4 \log y^{\frac{1}{2}} + \log \sqrt[3]{x}$

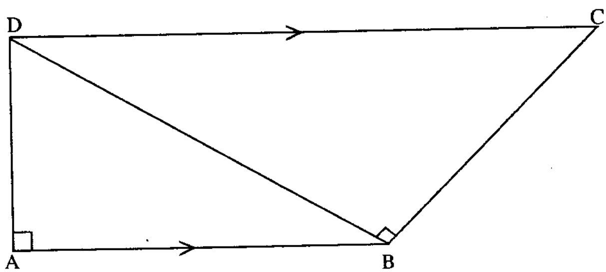
   (b) $\log x^4 - \frac{1}{4}\log y^3$

8. In the figure below, rectangles ABCD and PQRS are similar. Find the area of PQRS.



9. Given that $\log y^3 = \log \sqrt[3]{81}$, find y correct to 3 significant figures.

10. In the figure below, AB//DC, $\angle DAB = \angle DBC = 90^0$, CD = 7 cm and BC = 4 cm. Calculate the length of AB correct to four significant figures.



11. On a map with a scale of 1: 50 000, a coffee plantation covers an area of 20 cm^2. Find the area of the plantation in hectares.

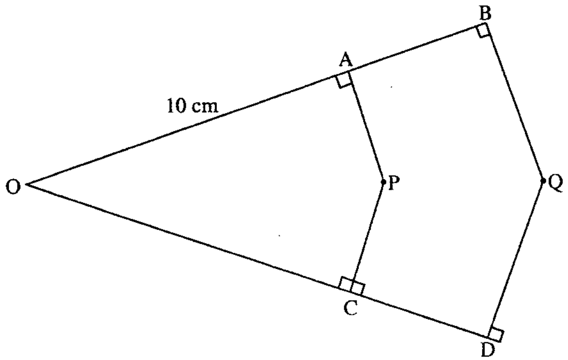
12. Determine the equation of the straight line with:

   (a) gradient $\frac{4}{3}$ and y-intercept -2.

   (b) gradient $-\frac{2}{5}$ and x-intercept $\frac{1}{4}$.

In each case, give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

13. In the figure below, the radii of the circles with centres P and Q are 5 cm and 10 cm respectively. If OA = 10 cm, calculate:
(a) OB (b) AB (c) PQ



14. The gradient of a straight line l_1 passing through the points P(3, 4) and Q (a, b) is $-\frac{3}{2}$. A line l_2 is perpendicular to l_1 and passes through the points Q and R(2, –1). Determine the values of a and b.

15. Evaluate $\{ (3.2)^{\frac{1}{2}} - 4\}^{-2}$, giving your answer in standard form.

16. The scale of a map is 1:125 000. What is the actual distance in kilometres represented by 16.8 cm on the map?

17. A triangle with the vertices of P(3, 0), Q(5, 1) and R(4, 4) is enlarged. If the centre of enlargement is (1, 0), find the co-ordinates of the image of the triangle when the scale factor is:
(a) –2 (b) $\frac{1}{2}$

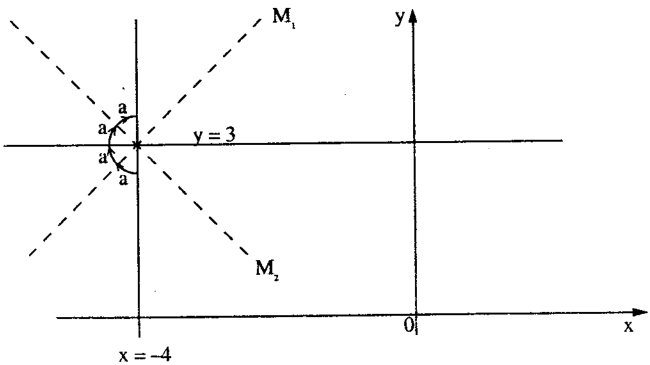
18. A line L is perpendicular to the $y = 3x$. If L passes through point (0, 4), find:
(a) the equation of L.
(b) the point Q where L intersects the line $y = 3x$.

19. If $A = \sqrt[3]{\frac{t\{W-B\}}{r}}$, find A if t = 0.0034, W = 4.634, B = 2.342 and r = 3.006.

20. The width of a rectangle is 10 cm and its area is 120 cm^2. Calculate the width of a similar rectangle whose area is 480 cm^2.
21. Three vertices of a rectangle ABCD are $A(3,-4)$, $B(7,-4)$ and $C(3,-2)$. The images of A and B under an enlargement are $A'(7,6)$ and $B'(3,6)$. Find:
    (a) the scale factor and centre of enlargement.
    (b) the image of C.
22. Two perpendicular lines intersect at (3, 9). If one of them passes through $(2, 9\frac{1}{3})$, find the equations of the two lines.
23. The volume of a sphere is given by $V = \frac{4\pi r^3}{3}$. Find the value of r if $v = 311$ and $\pi = 3.14$.
24. The vertices of a triangle are P(4, –3), Q(2, 1) and R(–8, 2). Find the gradients of PQ, QR and RP. Arrange the gradients in increasing order.
25. (a) A square whose vertices are P(1, 1), Q(2, 1), R(2,2) and S(1, 2) is given an enlargement with centre at (0, 0). Find the images of the vertices if the scale factors are:

    (i) 1 (ii) –1 (iii) 3 (iv) –3 (v) $\frac{1}{2}$ and (vi) $-\frac{1}{2}$

    (c) If the image of the vertices of the same square after an enlargement are P′(1. 1), Q′(5, 1), R′(5, 5) and S′(1, 5), find:
    (i) the centre, and,
    (ii) the scale factor of the enlargement.
26. The vertices of a quadrilateral are A(5, 1), B(6, 3), C(4, 4) and D(2, 3). It is reflected in the line $x = -4$, then in the line $y = x$, and finally in the y-axis. Find the co-ordinates of the final image.
27. In the figure below, A is the intersection point of $y = 3$ and $x = 4$, M_1 and M_2 are bisectors of the angles as shown:



(i) Write down the equations of M_1 and M_2.

(ii) Use M_1 and M_2 to show that vertically opposite angles are equal.

28. Given the co-ordinates of A and B as (4, 2) and (8, 2) respectively, find the equation of the perpendicular bisector of AB.

29. Find the co-ordinates of the vertices of the image of a kite whose vertices are P(0, 8), Q(3, 3) R(0, 1), and S(-3, 3) when rotated about the origin through:

(a) -90^0

(b) 180^0

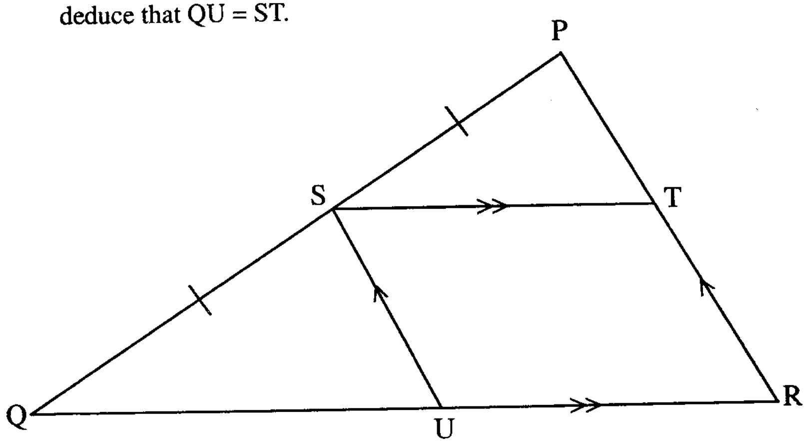
30. Plot the points A(4, –1), B(5, –3), C(4, –4), and D(3, –3) on a graph paper. Join all the points to form a polygon ABCD.

(a) What is the name of the polygon formed?

(b) Write down the equation of the mirror line of the polygon.

(d) Find the angle between AC and BD.

31. In the figure below, show that Δs PST and SQU are congruent. Hence, deduce that QU = ST.



32. An isosceles triangle ABC is right-angled at B. Points E and D on AC are such that AE = ED = DC. P is opposite B on the other side of AC such that PE = PE. Show that:

(a) triangles BEP and BDP are congruent.

(c) triangles AEP and CDP are congruent.

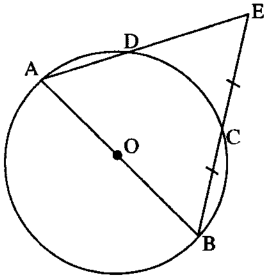
33. A triangle whose vertices are P(2, 2), Q(4, 2) and R(4, 4) is mapped onto a triangle whose vertices are P′(4, –2), Q′(2, –2) and R′(2, –4) under a rotation.

Find:

(a) the centre and angle of rotation.

(b) the images of points (0, 4) and (–1, 2) under the same rotation.

34. In the figure below, O is the centre of the circle. Chords AD and BC are produced to meet at E:



If BC = CE:

(a) show that AB = AE.

(b) show that triangles CDE and ABE are similar.

35. A figure has a rotational symmetry of order 4 about the point (6, –6). Two of its vertices are (8, –4) and (8, –8). Find the other vertices and draw the figures.

36. AOC and BOD are diameters of a circle ABCD, centre O. Show that:

(a) the Δs ACD and BDC are congruent.

(b) $\angle ACD = \angle BDC$.

37. Two circles with centres P and Q intersect at X and Y. XP and YQ are produced to meet the circles at A and B respectively. Show that AY is parallel to BX.

38. The ratio of the radii of two spheres is 2 : 3. Calculate the volume of the first sphere if the volume of the second is 20 cm^3.

39. Use mathematical tables to solve $0.1468^3 + \dfrac{1}{27.38^3}$.

40. Evaluate without using tables: $\sqrt[3]{\dfrac{0.729 \times 409.6}{0.1728}}$

41. A prism has a regular decagonal cross-section.

(a) How many axes of symmetry has the solid?

(b) What is the order of its rotational symmetry?

(c) How many planes of symmetry has the prism?

Chapter Eight

THE PYTHAGORAS' THEOREM

Consider the right-angled triangle ACB below:

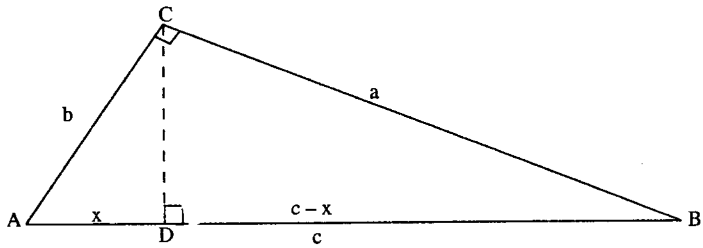


Fig. 8.1

Triangles BCD, ACD and BAC are similar, see below:

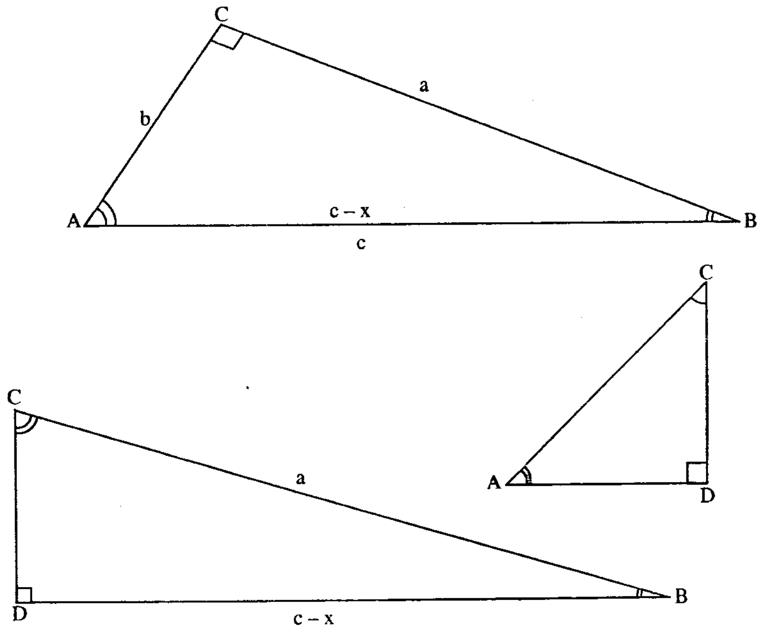


Fig. 8.2

$$\frac{BC}{BA} = \frac{BD}{BC}$$

$$\frac{a}{c} = \frac{c-x}{a}$$

$$a^2 = c(c-x)$$

Also, $\dfrac{AC}{BA} = \dfrac{AD}{AC}$

$$\frac{b}{c} = \frac{x}{b}$$

$$b^2 = cx$$

Adding the two;

$a^2 + b^2 = c(c - x) + cx$

$a^2 + b^2 = c^2 - cx + cx$

$a^2 + b^2 = c^2$

This is the pythagoras' theorem, which states that for a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the two shorter sides.

Example 1

In a right-angled triangle, the two shorter sides are 6 cm and 8 cm. Find the length of the hypotenuse.

Solution

Using Pythagoras' theorem;

$Hyp^2 = 6^2 + 8^2$

$Hyp^2 = 36^2 + 64$

$Hyp^2 = 100$

$Hyp = 10$ cm

Exercise 8.1

1. Figure 8.3 shows a triangle ABC which is right-angled at C.

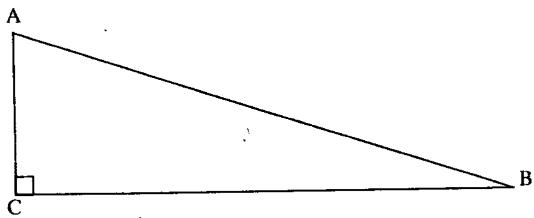


Fig. 8.3

Find:

(a) AB when AC = 5 cm and BC = 12 cm.
(b) AC when AB = 20 cm and BC = 16 cm.
(c) BC when AC = 10 cm and AB = 26 cm.
(d) AB when AC = 5 m and BC = 6 m.

2. Calculate the length of the diagonal of a rectangle whose sides are 6 cm and 8 cm long.

3. Find the length of the diagonal of a square of side 4 cm.

4. Find the height of an isosceles triangle if the equal sides are each 26 cm and the base is 48 cm long.

5. Find the length of a side of a square whose diagonal is 6 cm.

6. (a) Determine the length of a side of a rhombus whose diagonals are 18 cm and 24 cm long.
   (b) The shorter diagonal of a rhombus of side 25 cm is 30 cm. Find the length of the other diagonal.

7. What is the height of an equilateral triangle if one of its sides is 10 cm long?

8. Figure 8.4 shows a trapezium ABCD in which side AB is perpendicular to both AD and BC. Side AD = 17 cm, DC = 10 cm and CB = 9 cm. What is the length of side AB?

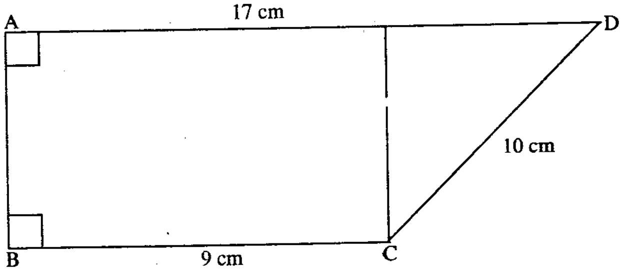


Fig. 8.4

9. A ladder 20 metres long leans against a building and reaches a point on the building that is 14 metres above the ground. How far from the bottom of the building is the foot of the ladder?

10. An electric pole is supported to stand vertically by a tight wire as shown in figure 8.5.

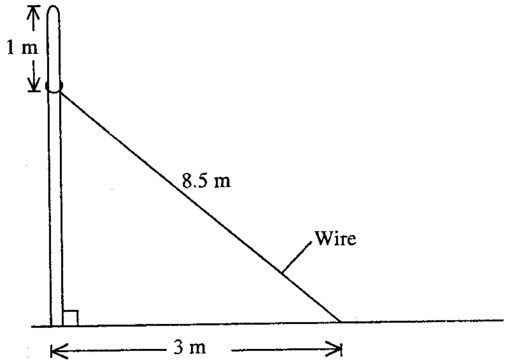


Fig. 8.5

Find the height of the pole above the ground.

11. In figure 8.6, BC = 24 cm, $\angle APD = \angle PCD = \angle PBA = 90^0$.

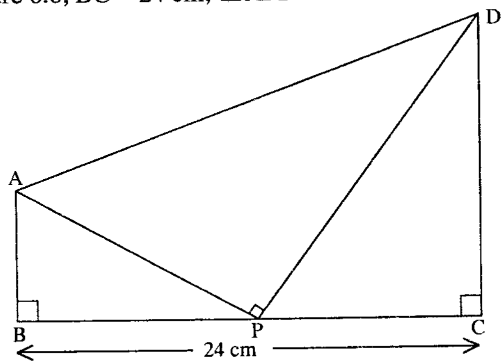


Fig. 8.6

If P is the midpoint of BC, find AD.

12. Figure 8.7 shows a triangle ABC which is right-angled at C. CB = 8 cm and AC = 6 cm. Find CD.

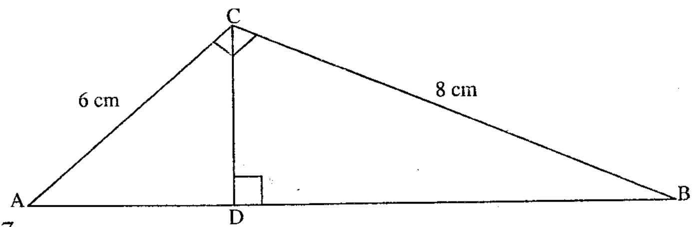


Fig. 8.7

Chapter Nine

TRIGONOMETRIC RATIOS

9.1: Tangent of an Acute Angle

The angle of elevation of the top of a flagpost 9 m high as viewed by an observer 10 m away from the post can be found by scale drawing, as shown in figure 10.1. In the figure, the angle of elevation is marked x.

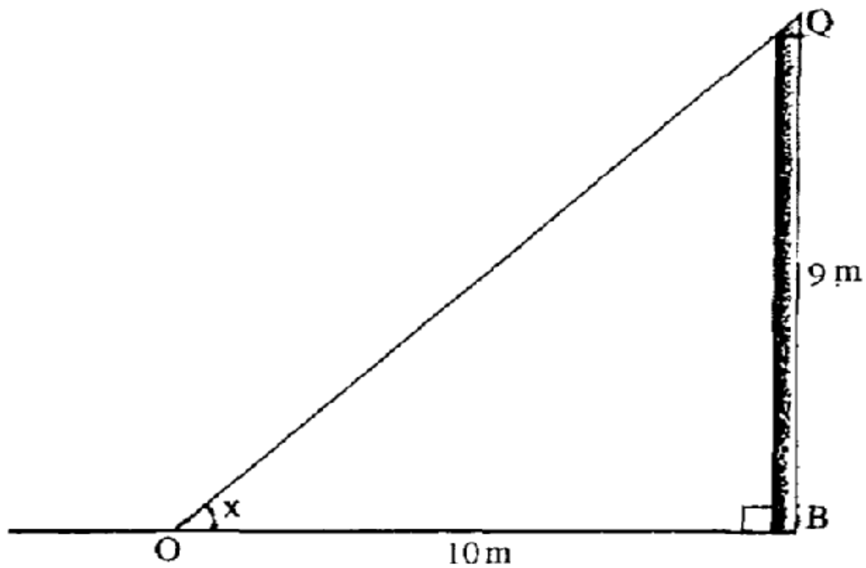


Fig. 9.1

In figure 9.2, OB is produced to V and OQ to T. A and C are two points on OV and P and R are points on OT such that AP and CR are parallel to BQ.

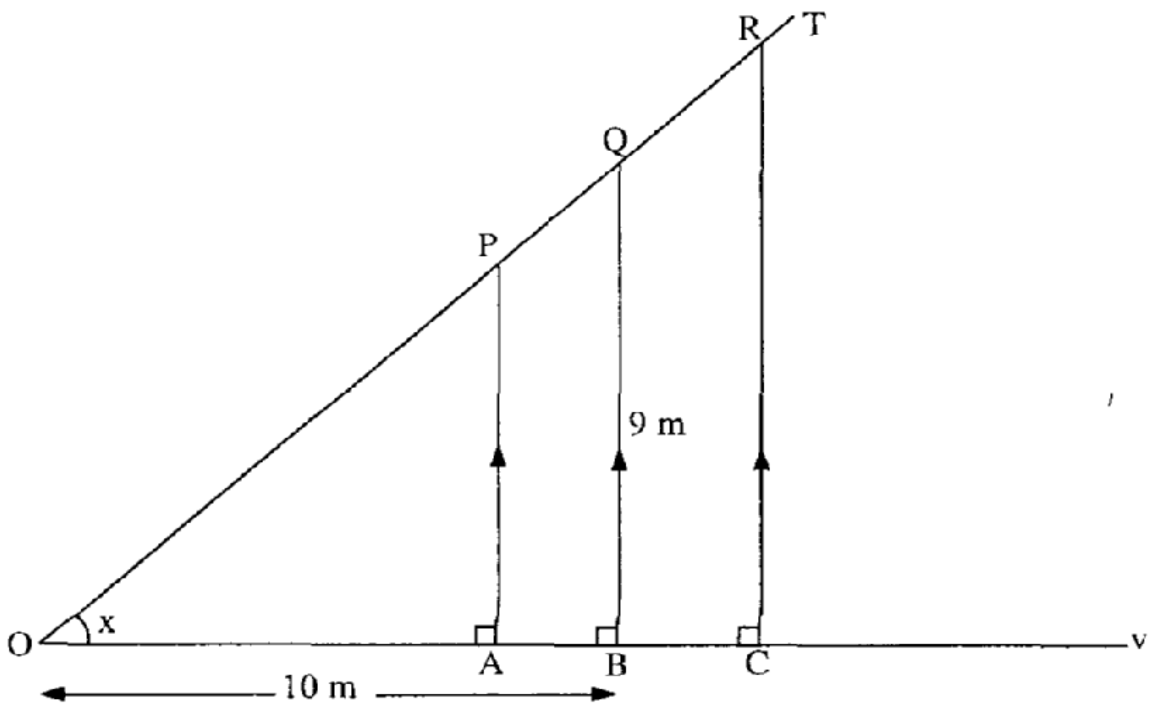


Fig. 9.2

Triangles OPA, OQB and ORC are similar. Therefore, $\frac{PA}{OA} = \frac{QB}{OB} = \frac{RC}{OC} = \frac{9}{10} = 0.9$.

By drawing any lines parallel to BQ, the ratio $\frac{\text{vertical distance}}{\text{horizontal distance}}$ will be the same for each triangle, that is 0.9.

This constant ratio, $\frac{\text{vertical distance}}{\text{horizontal distance}}$ is called the **tangent** of angle TOV.

Therefore, tangent of $\angle x = 0.9$

Note:

Tangent is abbreviated as 'tan'.

Therefore, tangent of $x = 0.9$ is written as $\tan x = 0.9$

The tangent of an angle depends on the size of the angle only. Figure 9.3 shows a right-angled triangle ABC, in which $\angle CAB = \theta$.

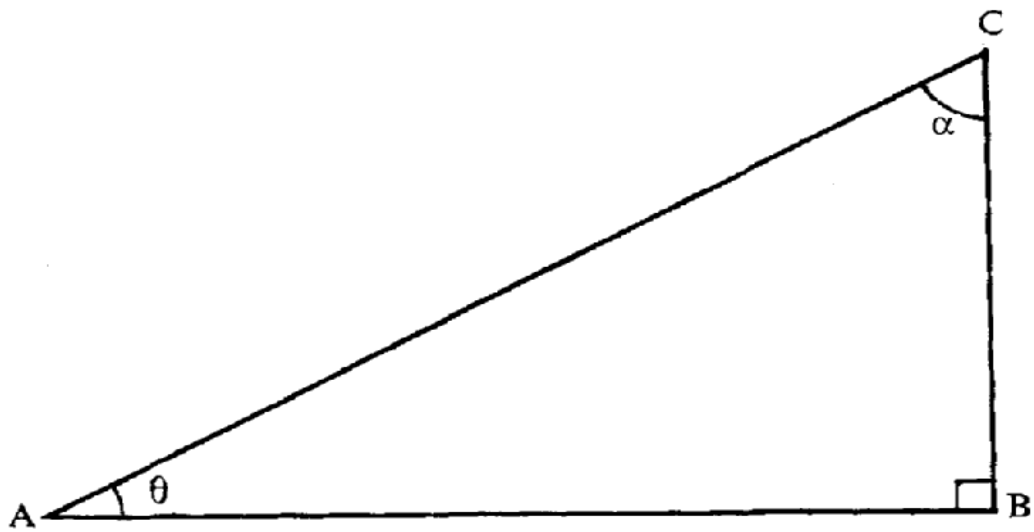


Fig. 9.3

The vertical side CB is opposite angle θ. The horizontal side AB is adjacent to angle θ.

In this case, $\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{CB}{AB}$

Similarly, express $\tan\alpha$ in terms of the lengths of the sides of the triangle.

Exercise 9.1

1. Identify opposite and adjacent sides of the marked acute angles in figure 9.4:

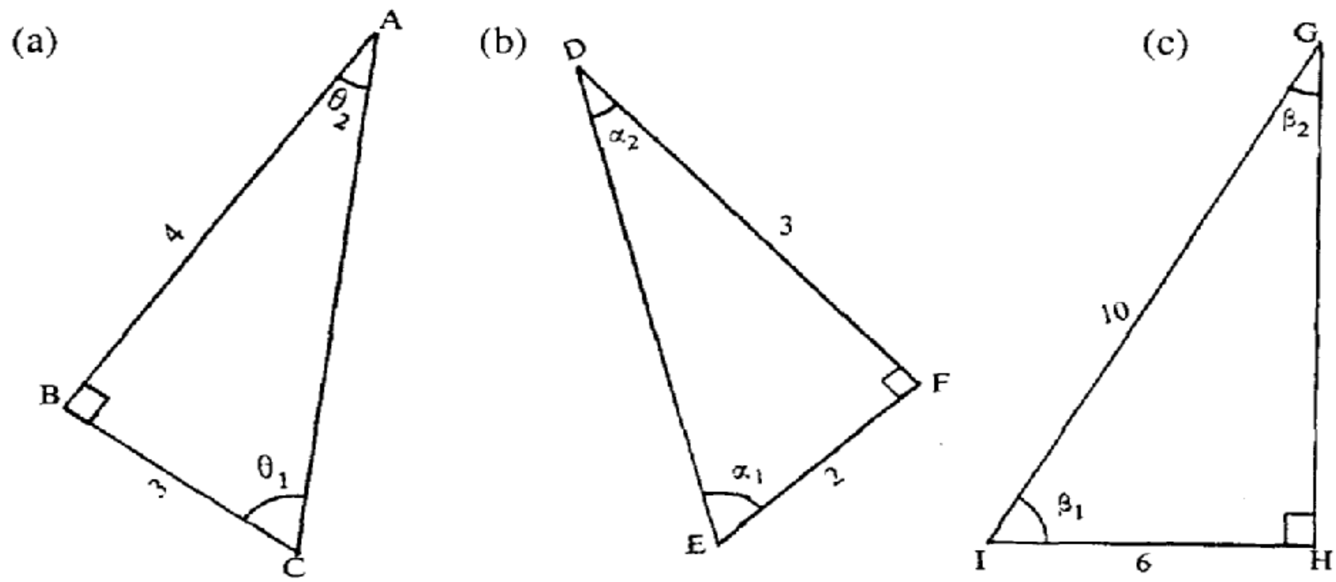
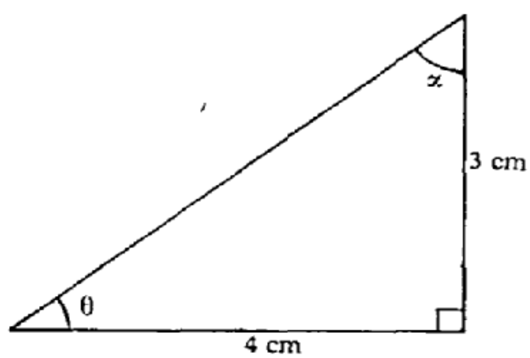


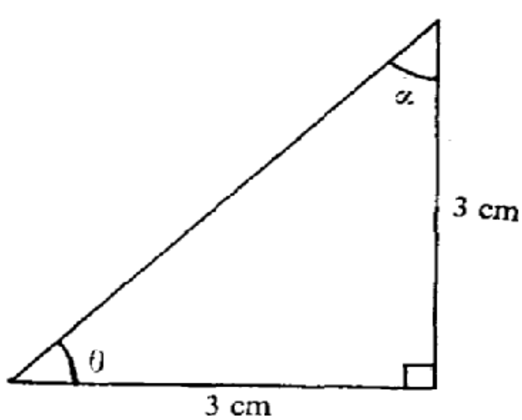
Fig. 9.4

2. Find the tangents of the marked acute angles in figure 9.4.
3. Draw the following triangles accurately. Using your diagram:
   (a) measure the indicated angles in degrees.
   (b) find the tangents of the indicated angles using the given measurements.

(i)



(ii)



(iii)

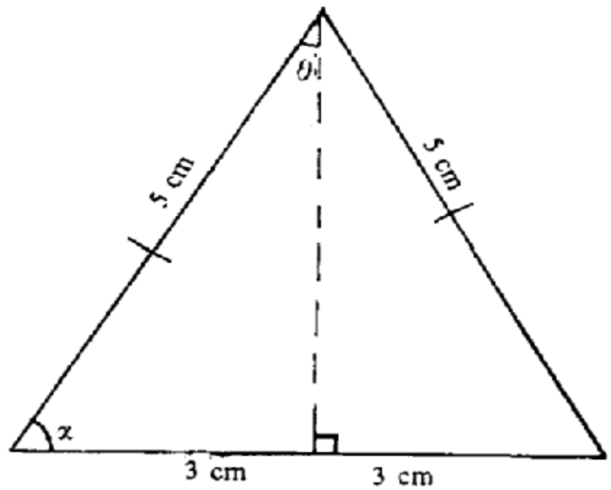


Fig. 9.5

9.2: Table of Tangents

Figure 9.6 shows a right-angled triangle ABC in which AB = 10 cm and $\angle CAB = 35^\circ$. By scale drawing, it is found that BC = 7 cm and therefore $\tan 35^\circ = 0.7$.

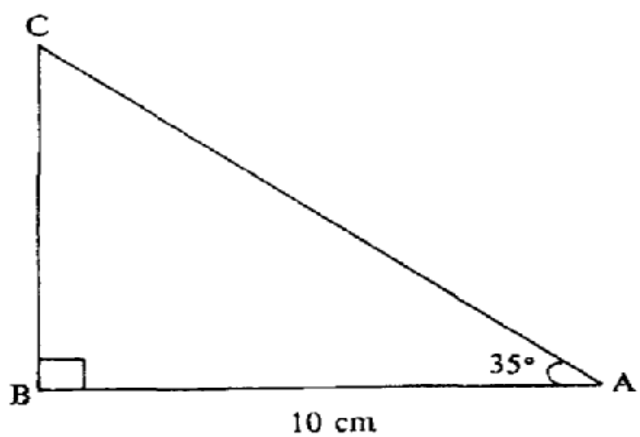


Fig. 9.6

Scale drawing is one method of obtaining the tangent of an angle. Special tables have been prepared and can be used to obtain tangents of acute angles (see tables of natural tangents in your mathematical tables). The technique of reading tables of tangents is similar to that of reading tables of logarithms or square roots.

(i) In the tables of tangents, the angles are expressed as decimals and degrees, or in degrees and minutes.

(ii) One degree is equal to $60'$ (60 minutes). Thus, $30' = 0.50^0$, $54'' = 0.9^0$ and $6' = 0.1^0$.

Note:

From the table, the values of tangents increase abruptly as the angles approach 90^0.

Example 1

Find the tangent of each of the following angles from the tables:

(a) 42^0 (b) 42.75^0 (c) $42^0\ 47'$

Solution

From the tables:

(a) $\tan 42^0 = 0.9004$

(b) Using decimal tables, $\tan 42.7 = 0.9228$. From the difference column under 5, we read 0.0016.

Therefore, $\tan 42.75^0 = 0.9228 + 0.0016$
$= 0.9244$

(c) Using degrees and minutes tables, $\tan 42^0\ 42' = 0.9228$ and from difference column under 5, we read 0.0027.

Therefore, $\tan 42^0\ 47' = 0.9228 + 0.0027$
$= 0.9255$

Example 2

In ΔPQR, $\angle PRQ = 90°$, $\angle PQR = 36°$ and $RQ = 4$ cm. Calculate PR.

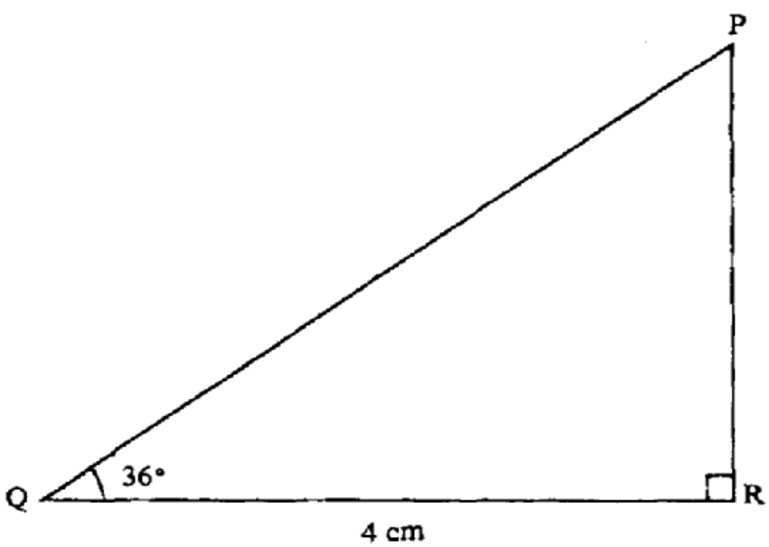


Fig. 9.7

Solution

$$\tan 36^0 = \frac{\text{opp}}{\text{adj}} = \frac{PR}{4}$$

$4 \tan 36^0 = PR$

Therefore, $PR = 4 \times 0.7265$
$= 2.9060$ cm

Example 3

In figure 9.8, AB = 5 cm, BC = 12 cm and $\angle ABC = 90^0$. Find x.

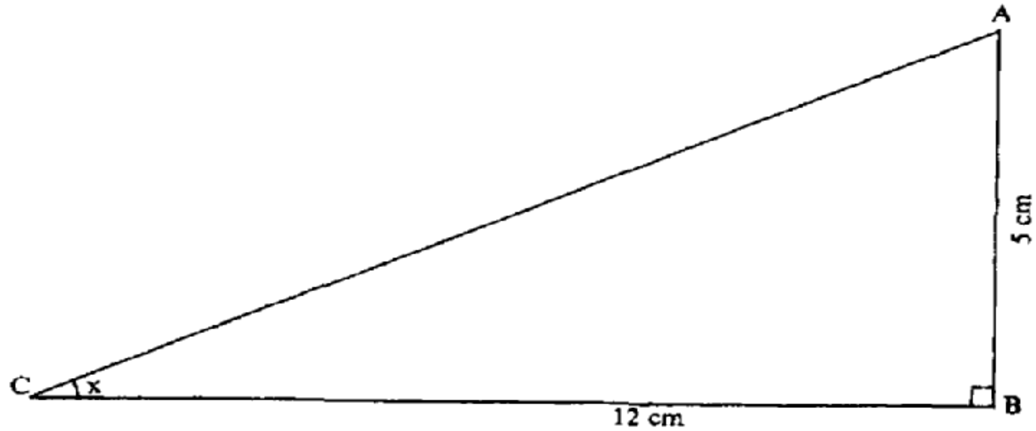


Fig. 9.8

Solution

$\text{Tan } x = \frac{AB}{CB} = \frac{5}{12} = 0.4167$

We notice that 0.4167 cannot be read directly from the tables of tangents. Therefore we look for a number nearest to 0.4167 from the tables. In this case, the nearest number is 0.4163. The angle whose tangent is 0.4163 is $22^0\,36'$

Between 0.4167 and 0.4163, we have a difference of 4. From the table of difference columns on the right hand side, the nearest number to 4 is 3, which gives a difference of $1'$. Adding $1'$ to $22^0\,36'$ we obtain $22^0\,37'$.

Therefore, the angle whose tangent is 0.4167 is $22^0\,37'$.

Thus, $x = 22^0\,37'$

Example 4

In figure 9.9, ABCD is a rhombus whose diagonals BD and AC intersect at O. If BD = 10 cm and AC = 6 cm, find the angles of the rhombus:

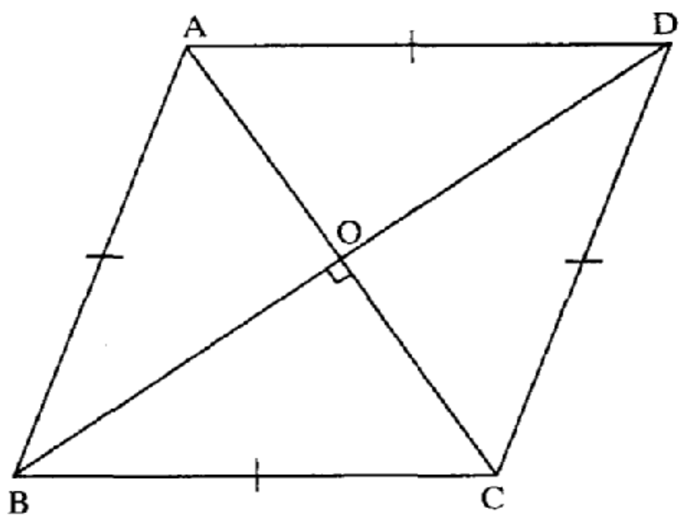


Fig. 9.9

Solution

ΔBOC in figure 9.10 is an extract from figure 9.9.

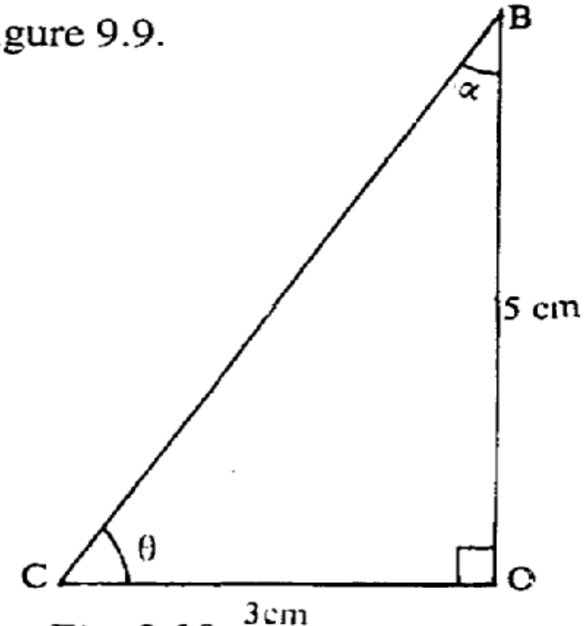


Fig. 9.10

$\text{Tan } \theta = \frac{5}{3} = 1.6667$

Therefore, $\theta = 59^0\, 2'$

$\text{Tan } \propto = \frac{3}{5}$

$= 0.6000$

$\propto = 30^0\ 58'$

$= 30.96^0$

Therefore, in figure 9.9;

$\angle BCD = 59^0\, 2' \times 2$

$= 118^0\, 4'$

$\angle CBA = 30^0\, 58' \times 2$

$= 61^0\, 56'$

$= 61.92^0$

and $\angle BCD = \angle BAD = 118^0\, 4'$

$\angle CBA = \angle ADC$

$= 61^0\, 56'$

$= 61.92^0$

Exercise 9.2

1. Express each of the following in degrees and minutes:

   (a) 15.3^0 (b) 25.75^0 (c) $30\frac{1}{2}^0$ (d) $34\frac{3}{4}^0$

2. Read from tables the tangent of:

   (a) 70.53^0 (b) 18.73^0 (c) $55^0\, 53'$ (d) $63^0\, 12'$
   (e) $81^0\, 08'$ (f) $10^0\, 30'$ (g) $75^0\, 58'$ (h) $89^0\, 54'$
   (i) $29^0\, 34'$ (j) $87^0\, 50'$ (k) $78^0\, 08^0$ (l) $48^0\, 42'$
   (m) 84^0 (n) $43^0\, 51'$ (p) 57.17^0

3. Find from the tables the angle whose tangent is:

   (a) 0.3317 (b) 0.6255 (c) 1.6391 (d) 0.44444
   (e) 0.0122 (f) 0.8799 (g) 0.1867 (h) 0.5903
   (i) 5.1006 (j) 1.0000 (k) 0.2839 (l) 2.0011
   (m) 3.6703 (n) 0.7400 (p) 40.92

4. A boy on top of a vertical wall 13.5 m high throws a ball down and notices that the ball hits a stone on the ground 18 m away from the foot of the wall. Calculate the angle of depression of the stone from the top of the wall.

5. A ladder leans against a wall so that its foot is 2.5 m away from the foot of the wall and its top is 4 m up the wall. Calculate the angle it makes with the ground.
6. A pedestrian notices that a tower is at a horizontal distance of 30 m away from him and that the angle of elevation of the top of the tower from where he is 35^0. Find the height of the tower.
7. A tree casts a shadow 20 m long. Find the height of the tree if the angle of elevation of the top of the tree from the tip of the shadow is 31^0.
8. In a right-angled triangle, the shorter sides are 4.5 cm and 9.2 cm long. Find the sizes of its acute angles.
9. One of the diagonals of a rhombus is 28 cm long and one of its angles is 70^0. Calculate the length of the second diagonal and hence the side of the rhombus **(two possible answers)**
10. From a window 25 m above a street, the angle of elevation of the top of a wall on the opposite side is 15^0. If the angle of depression of the base of the wall from the window is 35^0, find:
    (i) the width of the street.
    (ii) the height of the wall on the opposite side.
11. An aircraft flying into an airport calls out the control tower and says it is at height of 500 m above the tower. If its horizontal distance from the tower is 8 km, calculate its angle of elevation from the top of the tower.
12. Two buses X and Y are moving towards each other on a straight East-West main road. At a particular instant, another bus Z on another road is 50 km South of the main road and the bearings of X and Y from Z are 330^0 and 360^0 respectively. How far apart are X and Y?
13. ABC is a right-angled triangle inscribed in a circle ABC with AC as the diameter, see figure 9.11. If BC = 25 cm and AB = 40 cm, calculate angles BAC and ACB.

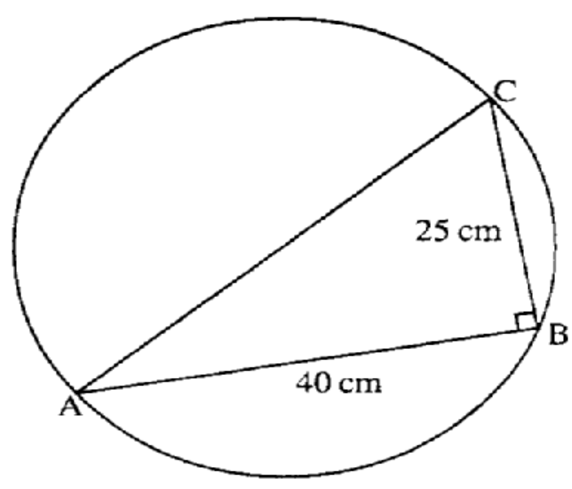


Fig. 9.11

14. Figure 9.12 shows five towns A, B, C, D and X connected by straight roads. BX = 40 km and XD = 48 km. If $\angle AXB = 56^0$, calculate the distances AB and BC:

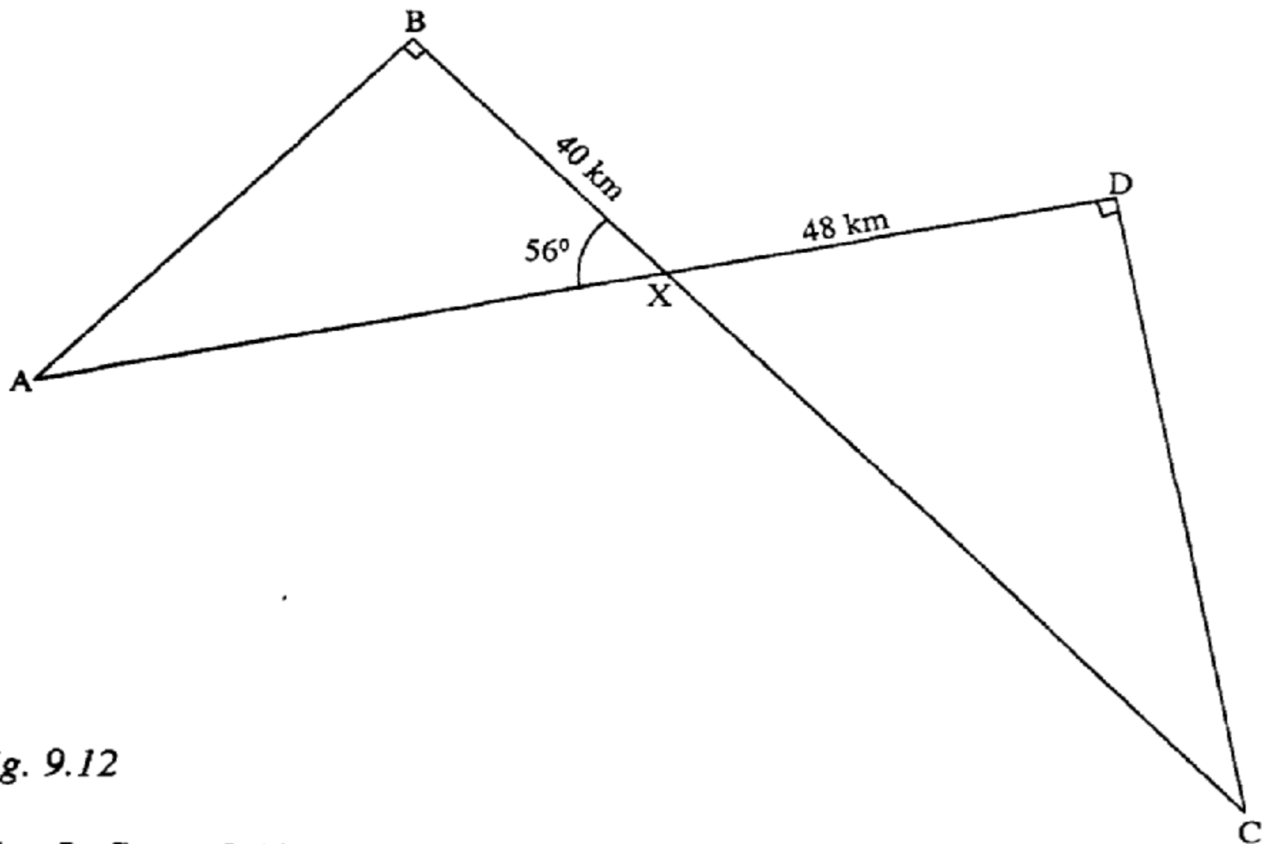


Fig. 9.12

15. In figure 9.13, calculate b, x, c and y in centimetres if a = 24 cm:

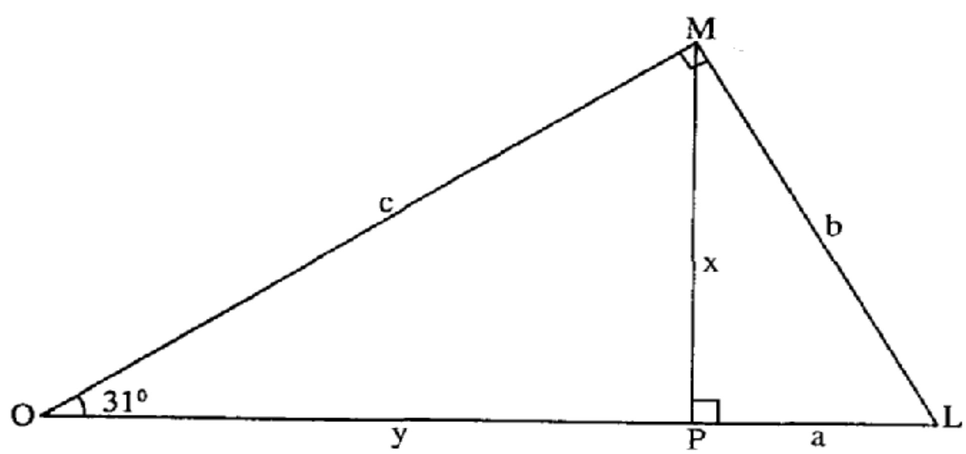


Fig. 9.13

16. Figure 9.14 shows a right-angled triangle ABC. AD is perpendicular to BC. If $\angle ABC = 30^0$ and AD = 6.5 cm, find the sides marked x, y, p and q.

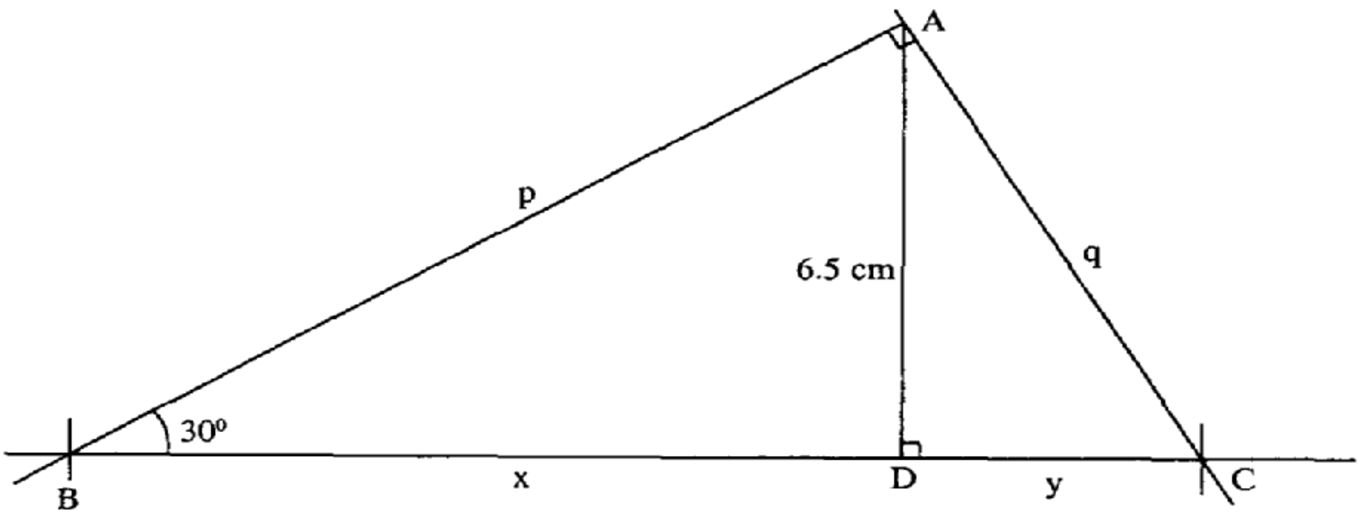


Fig. 9.14

9.3: Sine and Cosine of an Acute Angle

In figure 9.15, LA, MB and NC are perpendicular to OY and $\angle XOY = \theta$.

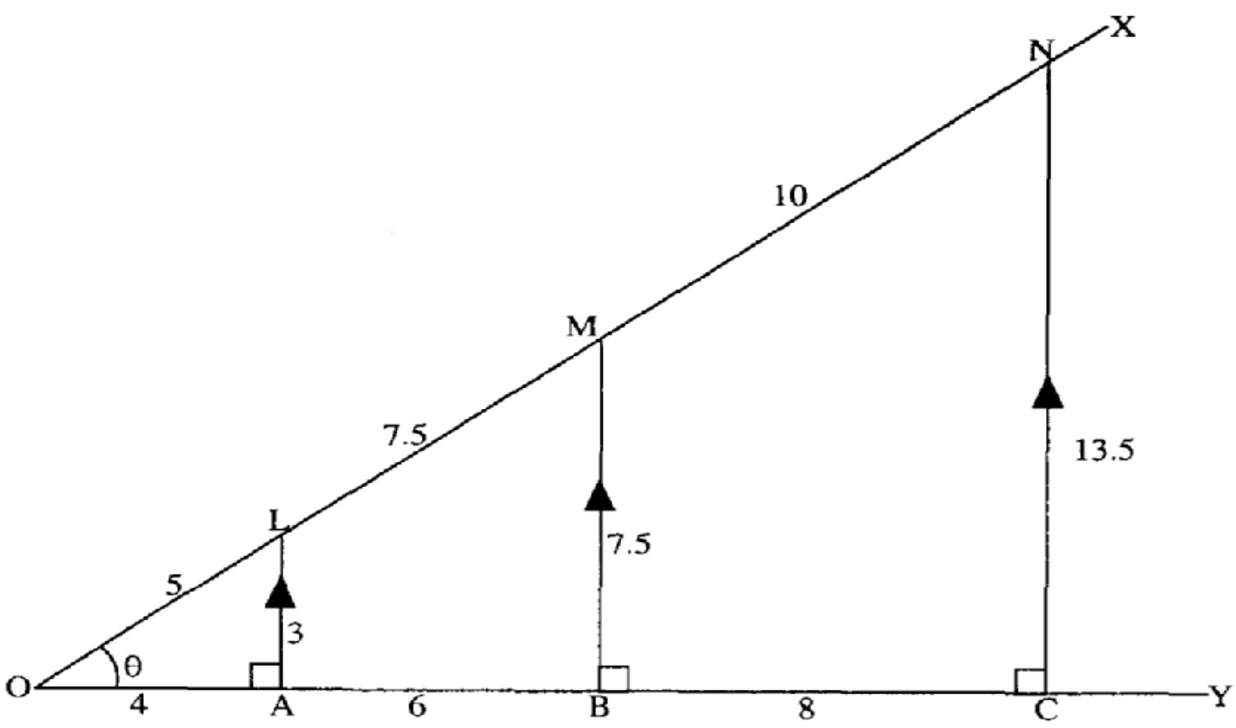


Fig. 9.15

(a) Copy the following and fill in the blank spaces:

(i) $\frac{AL}{OL} = \frac{3}{5}$ (ii) $\frac{BM}{OM} = \ldots\ldots$ (iii) $\frac{CN}{ON} = \ldots\ldots$

Note:

$$\frac{AL}{OL} = \frac{BM}{OM} = \frac{CN}{ON} = \frac{3}{5}$$

This constant value is obtained by taking the **ratio of the side opposite the angle θ to the hypotenuse side** in each case. This ratio is called the **sine** of angle θ, abbreviated as sin θ.

(b) Copy the following and fill in the blank spaces:

(i) $\frac{OA}{OL} = \ldots\ldots\ldots$ (ii) $\frac{OB}{OM} = \ldots\ldots\ldots$ (iii) $\frac{OC}{ON} = \ldots\ldots$

Note that $\frac{AO}{OL} = \frac{OB}{OM} = \frac{OC}{ON} = \frac{4}{5}$

This constant value is obtained by taking the **ratio of the side adjacent to angle θ to the hypotenuse** in each case. This constant ratio is called the **cosine** of angle θ, abbreviated as cos θ.

In general, given a right-angled triangle ABC as shown in figure 9.16 below:

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{AC}{AB}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{BC}{AB}$$

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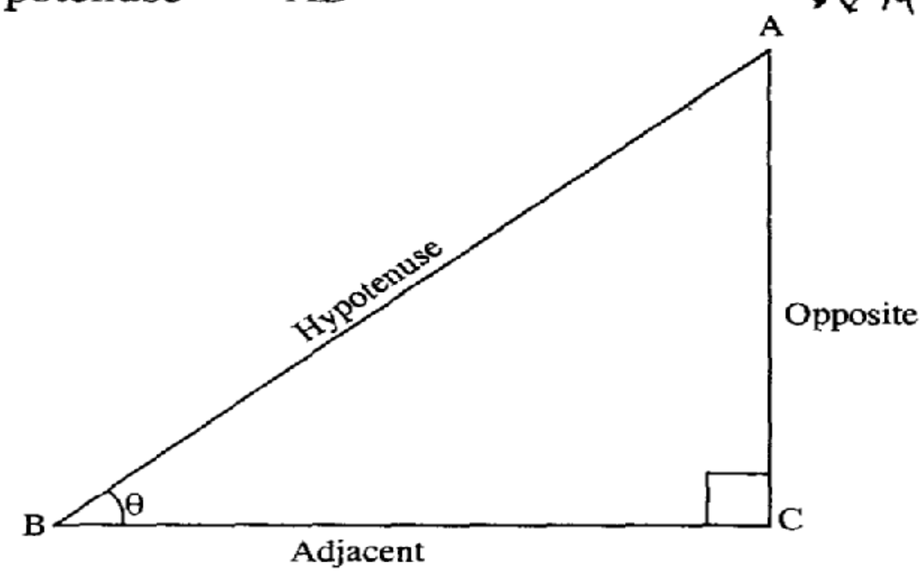


Fig. 9.16

Example 5

In figure 9.17, AB = 5 cm, CB = 12 cm and $\angle ABC = 90^0$. Calculate:

(a) sin x.

(b) cos x.

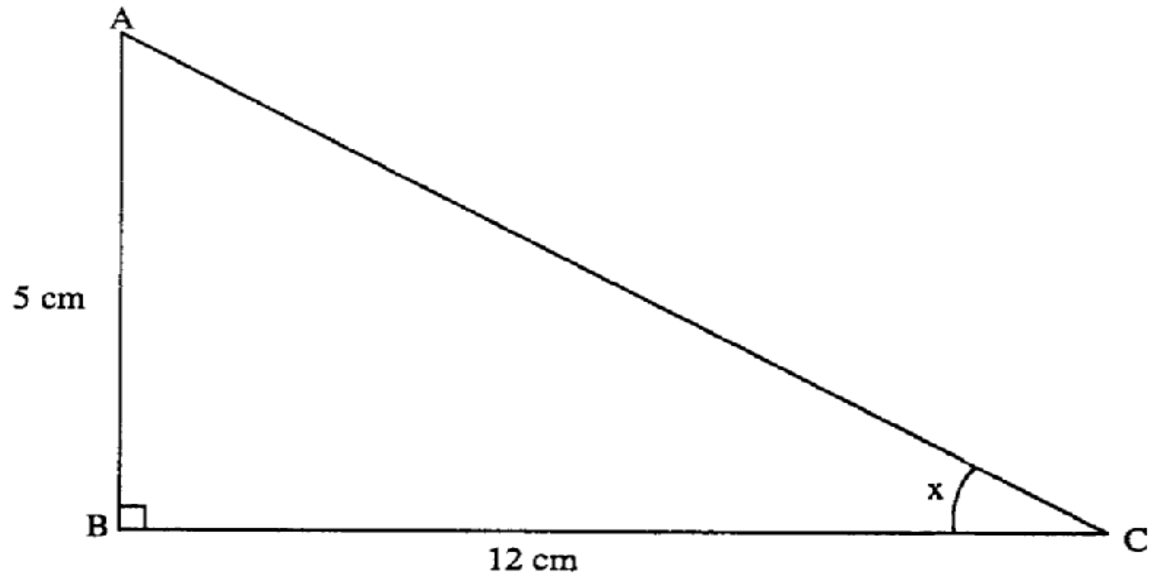


Fig. 9.17

Solution

(a) $\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{AB}{AC} = \frac{5}{AC}$

But $AC^2 = 12^2 + 5^2$

$= 144 + 25$

$= 169$

Therefore, $AC = 13$ cm

$\therefore \sin x = \frac{5}{13}$

$= 0.3846$

(b) $\cos x = \frac{\text{adj}}{\text{hyp}}$

$= \frac{CB}{AC}$

$= \frac{12}{13}$

$= 0.9231$

Example 6

Find each of the following by scale drawing:

(a) (i) $\sin 30^0$

(ii) $\cos 30^0$

(b) Angle whose (i) sine is 0.7

(ii) cosine is 0.9

Solution

(a)
- Draw a line AB of any length.
- Measure $\angle CAB = 30^0$ (see figure 9.18)
- From a point L on AC, draw a perpendicular to AB to meet AB at M. In this case, LM = 5.2 cm and AL = 10.4 cm.

$$\sin 30^\circ = \frac{LM}{AL} = \frac{5.2 \text{ cm}}{10.4 \text{ cm}} = 0.5$$

$$\cos 30^\circ = \frac{AM}{AL} = \frac{9.0}{10.4} \approx 0.9$$

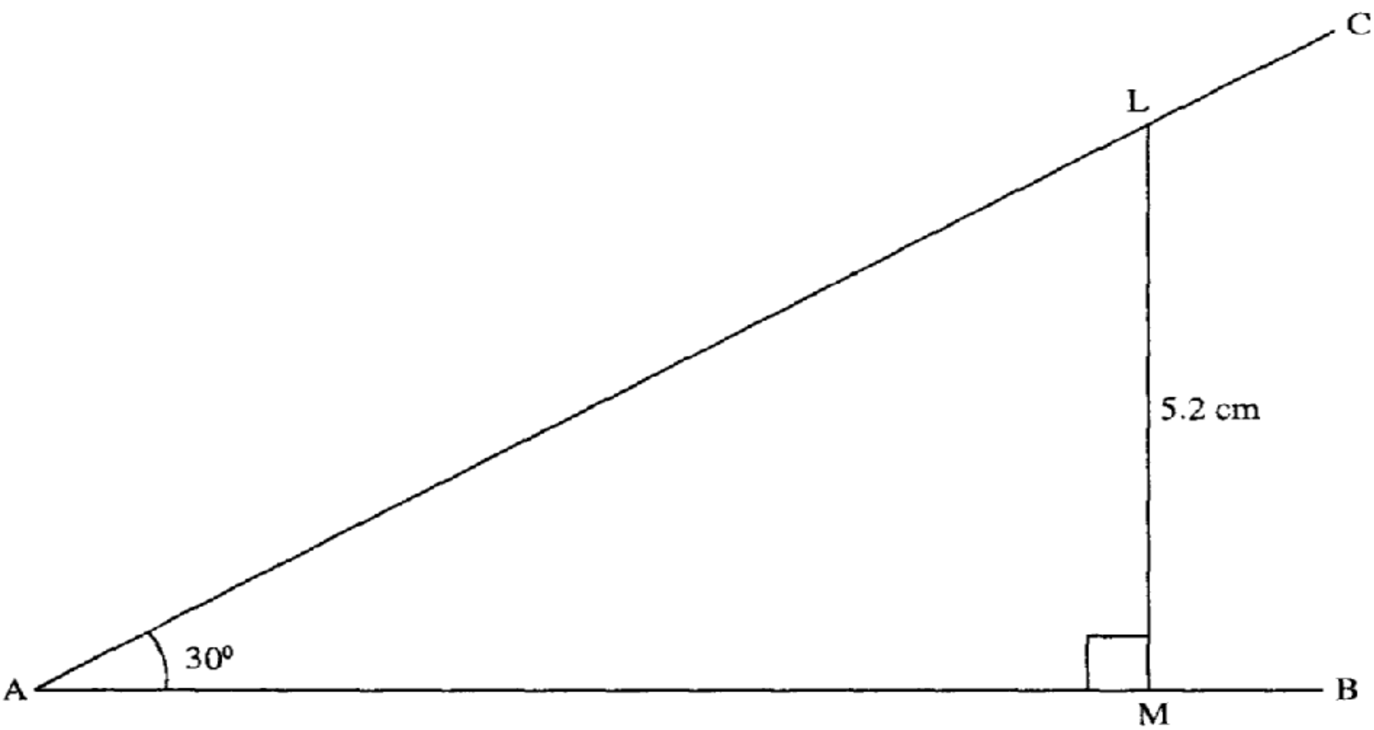


Fig. 9.18

Using the same method as in (a) find:

(i) $\sin 65^0$

(ii) $\cos 65^0$

(b) (i) $0.7 = \frac{7}{10}$
- Draw a line XY of any length.
- Construct a line LM, 7 cm long and perpendicular to XY at M.
- With centre L and a radius of 10 cm, draw an arc to cut XY at N.
- By measuring, $\angle MNL \approx 44.5^0$.
  Therefore, $\sin 44.5^0 \approx 0.7$

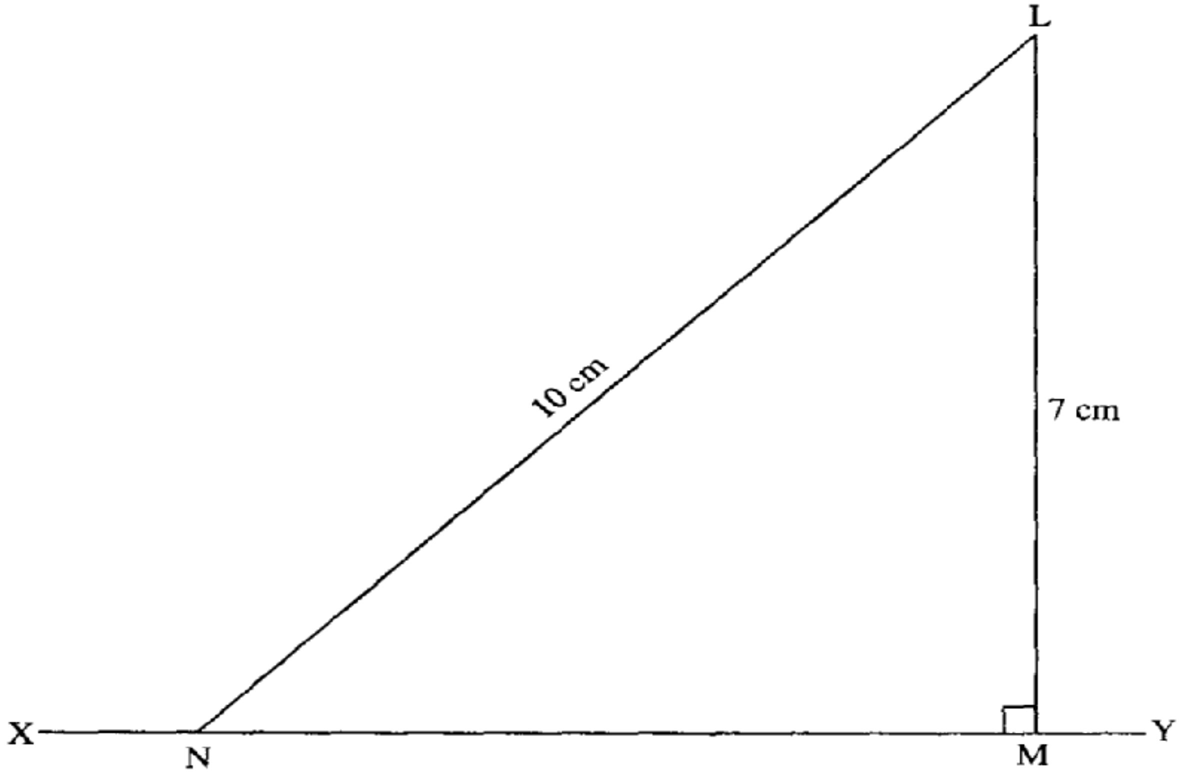
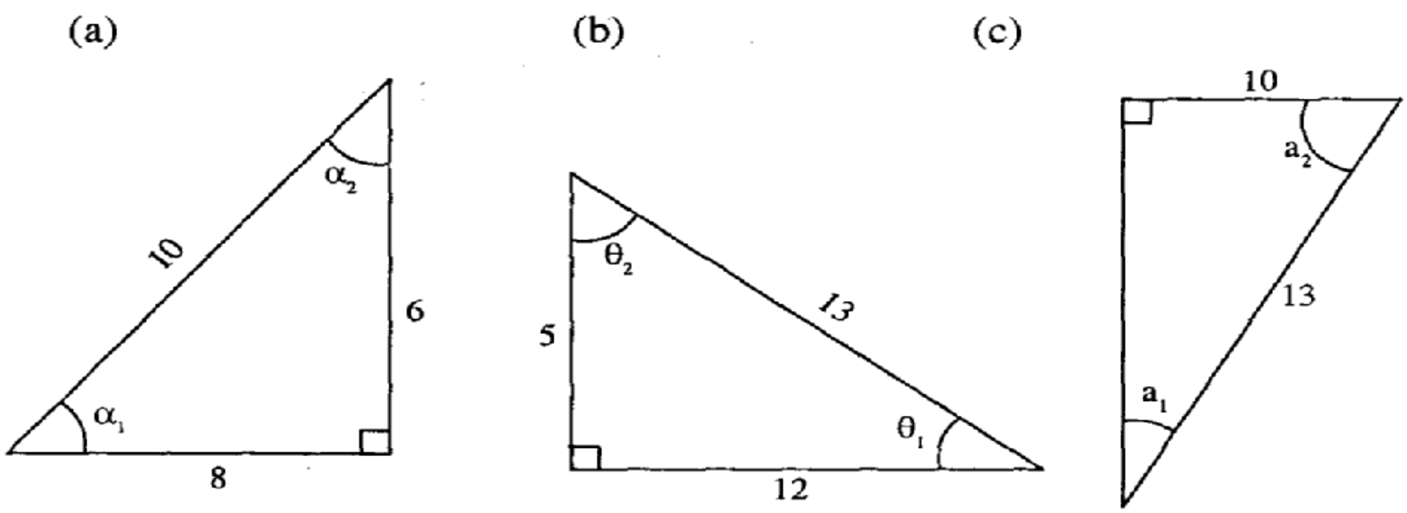


Fig. 9.19

Follow the same procedure to find the angle whose cosine is 0.9.

Exercise 9.3

1. Find the cosine and sine of each in the following marked angle. (Units are in centimetres)



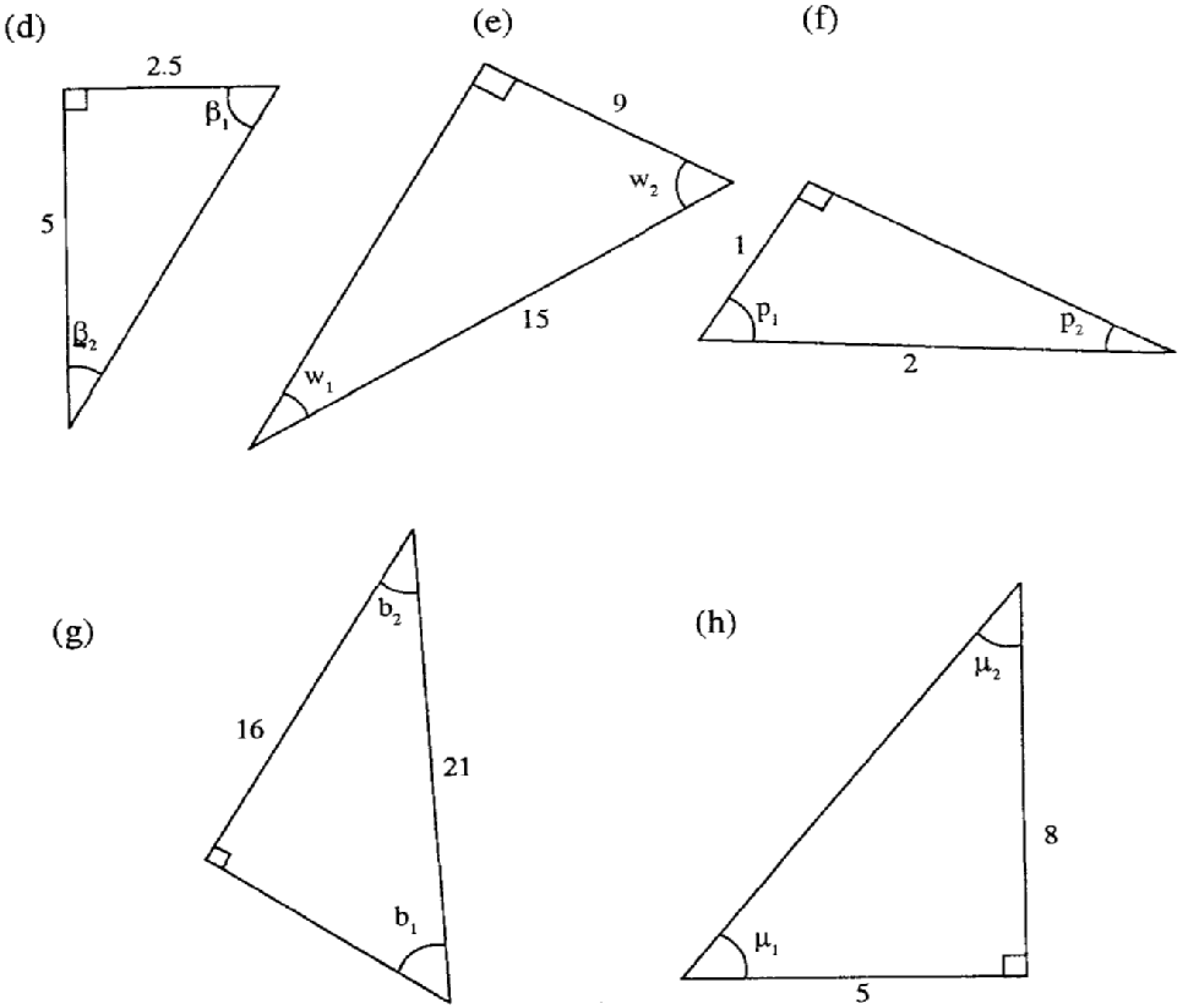


Fig. 9.20

2. (a) Find by scale drawing the angle whose sine is:

   (i) $\frac{7}{12}$ (ii) 0.5 (iii) 0.30

   (iv) 0.6 (v) 0.7 (vi) $\frac{3}{5}$

   (vii) $\frac{4}{7}$ (viii) $\frac{2}{5}$ (ix) $\frac{5}{6}$

   (b) Find by scale drawing the angle whose cosine is:

   (i) $\frac{1}{3}$ (ii) $\frac{7}{9}$ (iii) $\frac{1}{5}$ (iv) 0.6

   (v) $\frac{5}{13}$ (vi) $\frac{4}{5}$ (vii) $\frac{10}{17}$ (viii) 0.72

3. (a) If $\sin\theta = \frac{3}{5}$, find:
   (i) $\cos\theta$.
   (ii) $\tan\theta$.

   (b) If $\sin\theta = \frac{1}{2}$, find $\tan\theta$.

   (c) If $\tan\theta = \frac{1}{\sqrt{3}}$, find $\cos\theta$.

   (d) If $\cos\theta = \frac{1}{\sqrt{2}}$, find $\sin\theta$.

9.4: Tables of Sine and Cosine

The sine and cosine tables are read and used in the same way as the tangent tables. As the angles increase from 0^0 to 90^0:

(i) the values of their sines increase from 0 to 1.

(ii) the values of their cosines decrease from 1 to 0.

Therefore, the values in the difference columns of cosine tables have to be subtracted and those in the difference columns of the sine tables have to be added.

Example 7

Read the sine and cosine values of the following angles from the tables:

(a) 47.3^0

(b) 69.55^0

Solution

(a) $\sin 47.3^0 = 0.7349$
    $\cos 47.3^0 = 0.6782$

(b) $\sin 69.55^0 = 0.9370$
    $\cos 69.55^0 = 0.3494$

Example 8

Find x and y in figure 9.21 below:

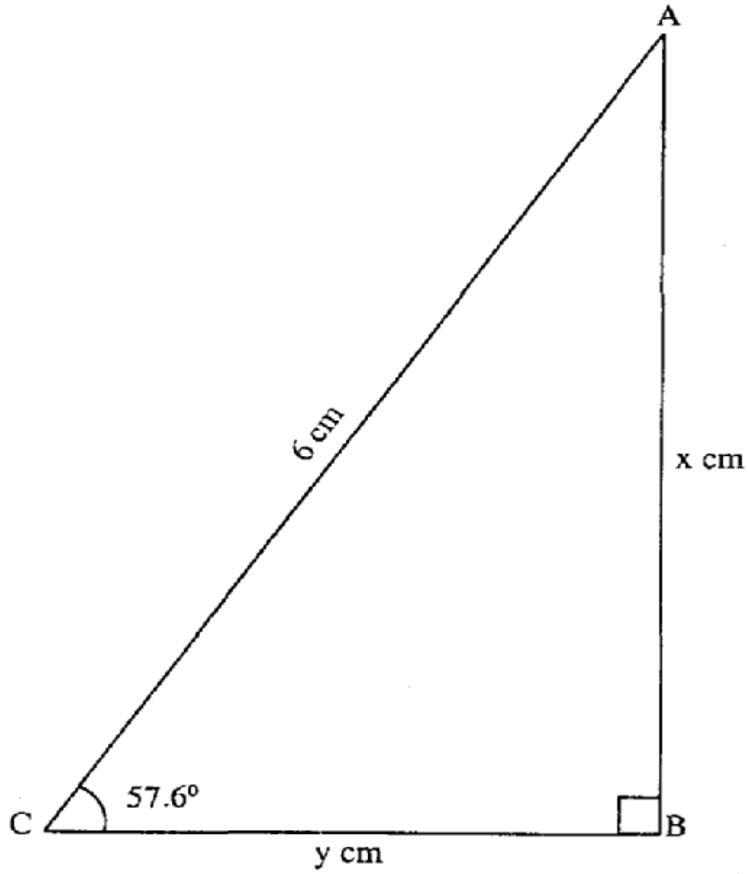


Fig. 9.21

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MWANEI

Solution

$\sin 57.6° = \frac{x}{6}$

$6 \sin 57.6° = x$

Therefore, $x = 6 \times 0.8443$

$= 5.0658$

$= 5.066$ (to 4 s.f.)

$\cos 57.6° = \frac{y}{6}$

$6 \cos 57.6° = y$

Therefore, $y = 6 \times 0.5358$

$= 3.2148$

$= 3.215$ (to 4 s.f.)

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Exercise 9.4

1. Find from tables the angle whose sine is:

(a) 0.3367	(b) 0.5871	(c) 0.0523	(d) 0.8500
(e) 0.1822	(f) 0.9834	(g) 0.5012	(h) 0.2518

2. Find from the tables the angle whose cosine is:

(a) 0.1643	(b) 0.7196	(c) 0.9970	(d) 0.8660
(e) 0.4009	(f) 0.9481	(g) 0	(h) 0.7371

3. Read from the tables the sine of:

(a) 31.46^0	(b) $77^0\,34'$	(c) $52^0\,9'$	(d) $66^0\,31'$
(e) 6.76^0	(f) 40.13^0	(g) $26^0\,47'$	(h) 13.07^0

4. Read from the tables the cosine of:

(a) $79^0\,42'$	(b) 24.23^0	(c) $5^0\,37'$	(d) 60^0
(e) $88^0\,59'$	(f) 55.97^0	(g) 33.33^0	(h) $17^0\,52'$

5. Find the values of the unknown side in each of the following figures:

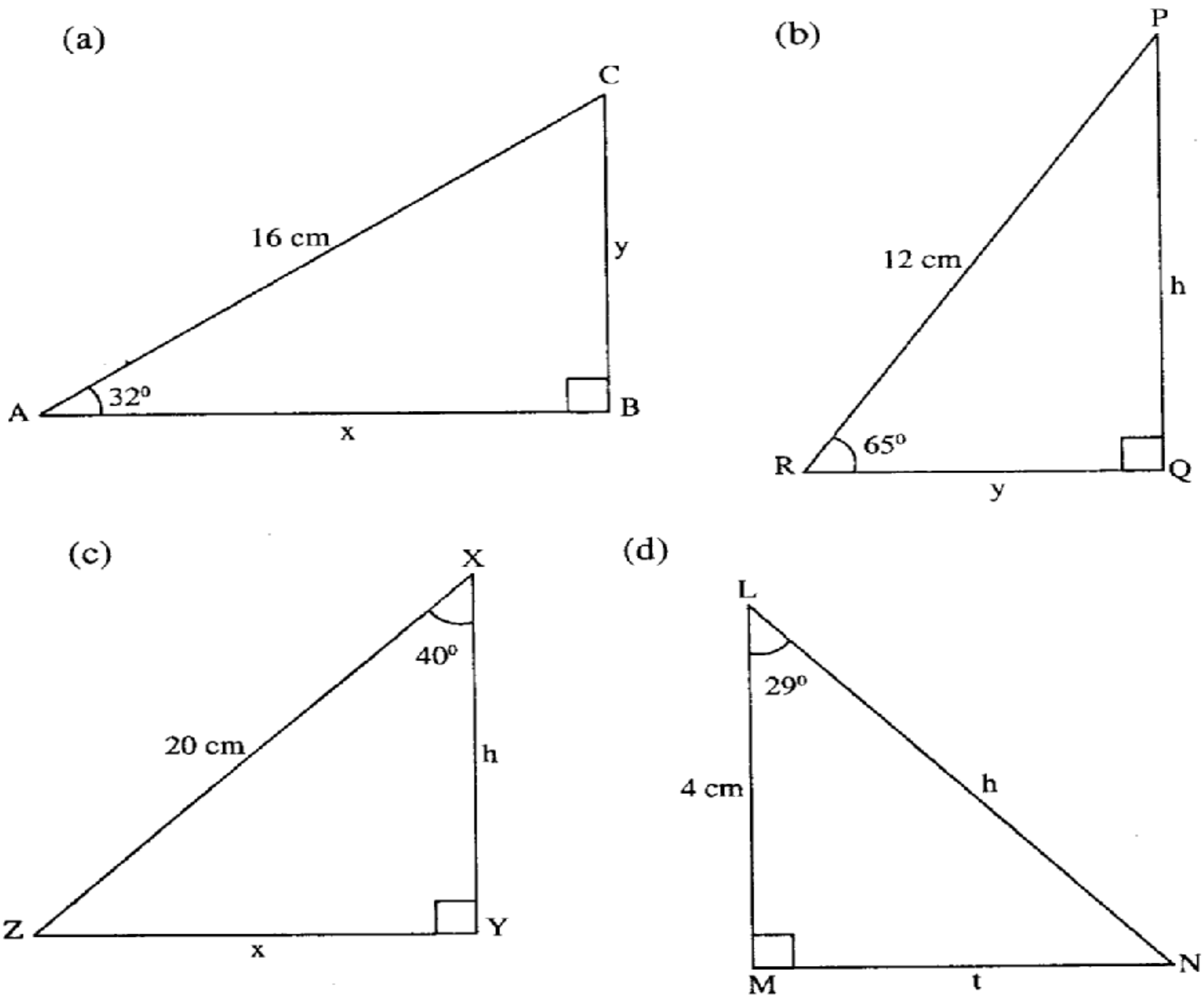


Fig. 9.22

6. PQRS is a rhombus of side 7 cm and $\angle PQR$ is 65^0. Calculate the lengths of PR and QS.

7. A ladder 10 m long leans against a wall as shown in figure 9.23. If its foot is 3 m from the wall, calculate:
   (a) the vertical distance from the top of the ladder to the ground.
   (b) the angle the ladder makes with the ground.

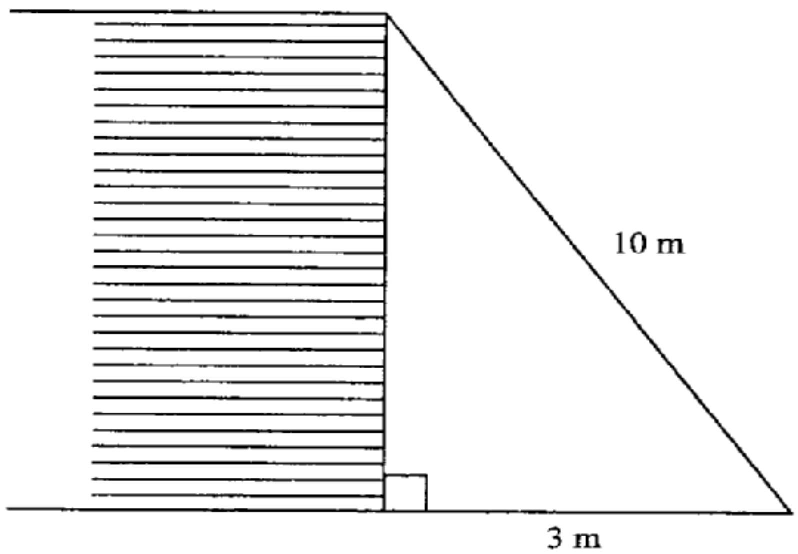


Fig. 9.23

8. From a point P on the ground, 20 m away from the foot of a building, the angle of elevation of the top of the building is 25^0. Find:
   (a) the height of the building.
   (b) the shortest distance to the top of the building from point P.
9. Figure 9.24 shows an isosceles triangle in which AB = AC = 6 cm. Angle BAC is 80^0. Calculate the length of BC:

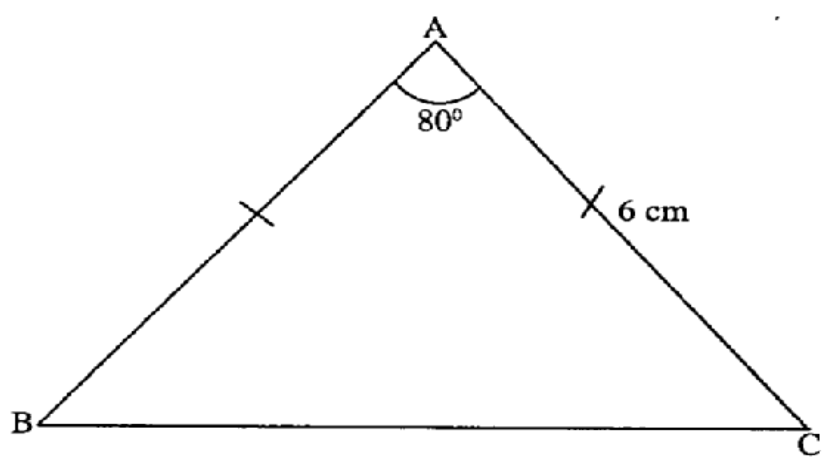


Fig. 9.24

10. The angle of depression of a car from the top of a building 8 m high is 44^o. How far is the car from the foot of the building?

11. Figure 9.25 shows a trapezium PQRS in which PQ = 7 cm, PS = 5 cm, $\angle PQR = \angle QRS = 90^0$ and $\angle SPQ = 125^0$. Calculate the lengths QR and RS.

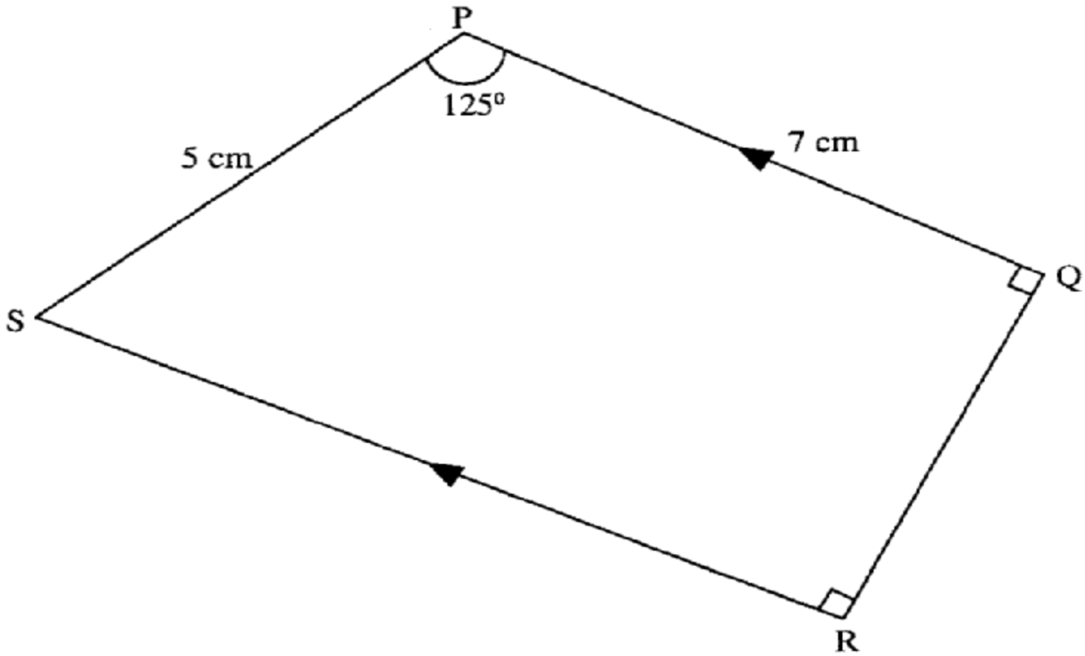


Fig. 9.25

12. In figure 9.26, PT = 3.2 cm, RS = 10 cm, $\angle SPT = 51^0$ and $\angle PRQ = 33^0$. Calculate lengths of the sides marked a, b, c, d and e.

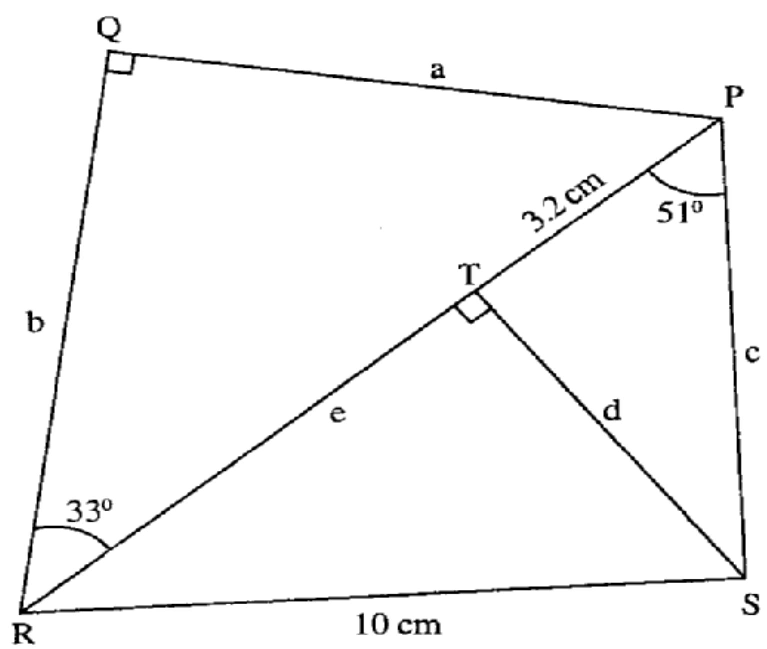


Fig. 9.26

13. Calculate the values of p, q, r and s in figure 9.27:

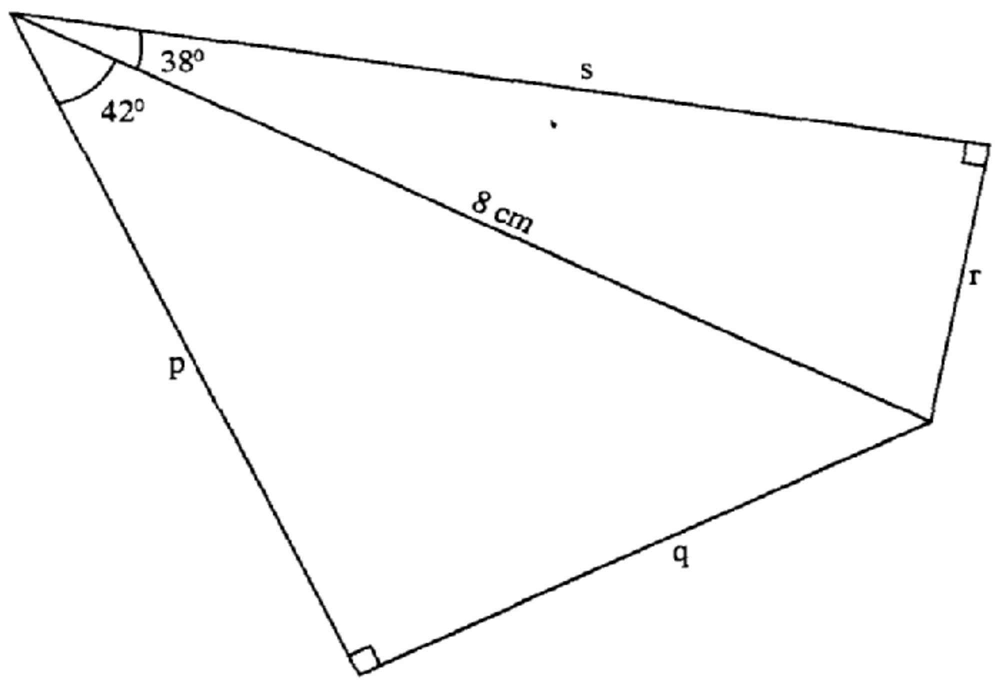


Fig. 9.27

14. The diagonals of a rectangle PQRS are 13 cm long. If $\angle SQR = 27^0$, find the dimensions of the rectangle.
15. In figure 9.28, WXYZ is a trapezium with WZ parallel to XY. WX =15.61 cm, XY = 32.73 cm, ZY = 9.73 cm, $\angle WXY = 26^0$ and $\angle ZYX = 43^0$. Calculate the length of WZ:

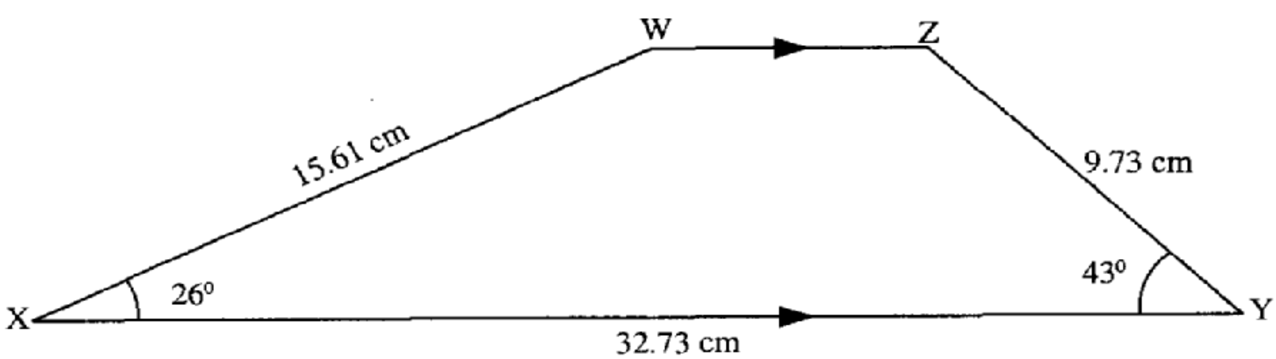


Fig. 9.28

16. In figure 9.29, AB = 8 cm, CD = 16 cm, $\angle ACD = 70^0$ and $\angle BAE = 50^0$. Calculate ED.

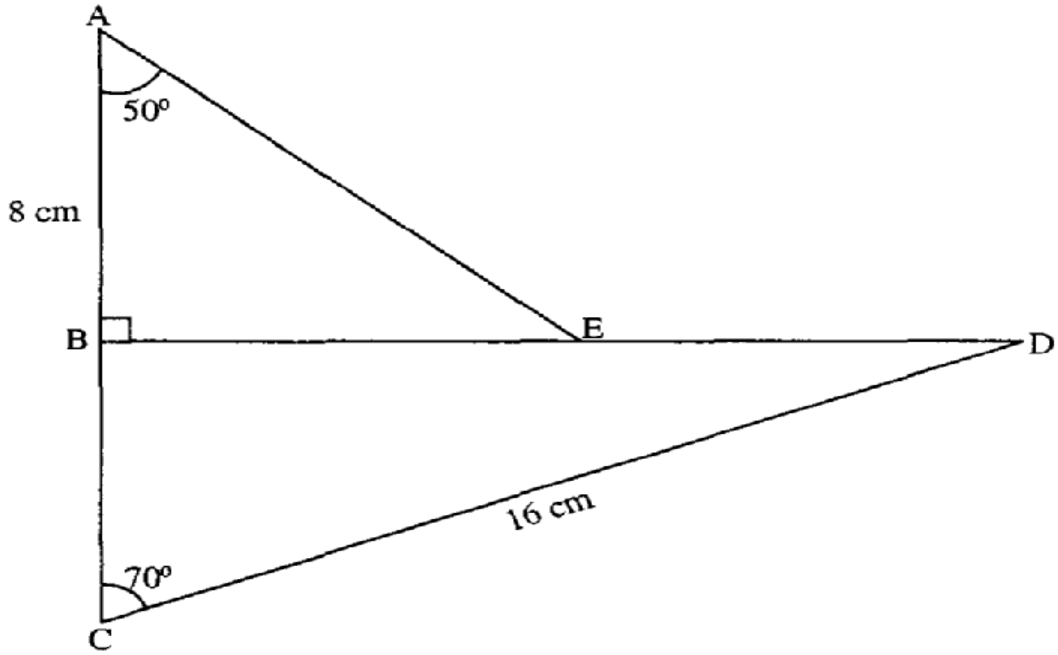


Fig. 9.29

17. The angle at the vertex of a pair of dividers is 54°. The tips of the arms are 12 cm apart. If the dividers are held upright, calculate:
    (a) the angle between the horizontal and the arms.
    (b) the length of each arm.
    (c) the height of the vertex above the horizontal.
18. A regular octagon of side 6 cm is inscribed in a circle. Calculate the diameter of the circle.
19. In figure 9.30, QS = 20 cm, TS = 7 cm, ∠RTS = 65° and ∠QPR = 25°. Find the length of PT.

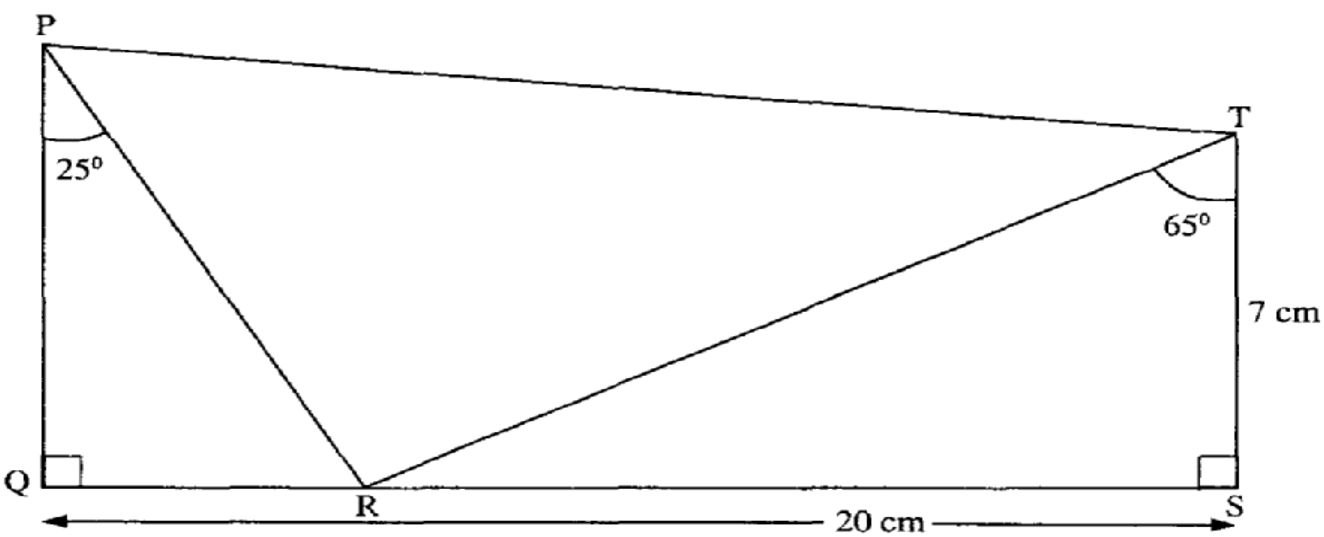


Fig. 9.30

20. In figure 9.31, LN = 7.6 cm, $\angle KLM = 38^0$, $\angle KMN = 53^0$ and $\angle KNL = 90^0$. Calculate KM and KL.

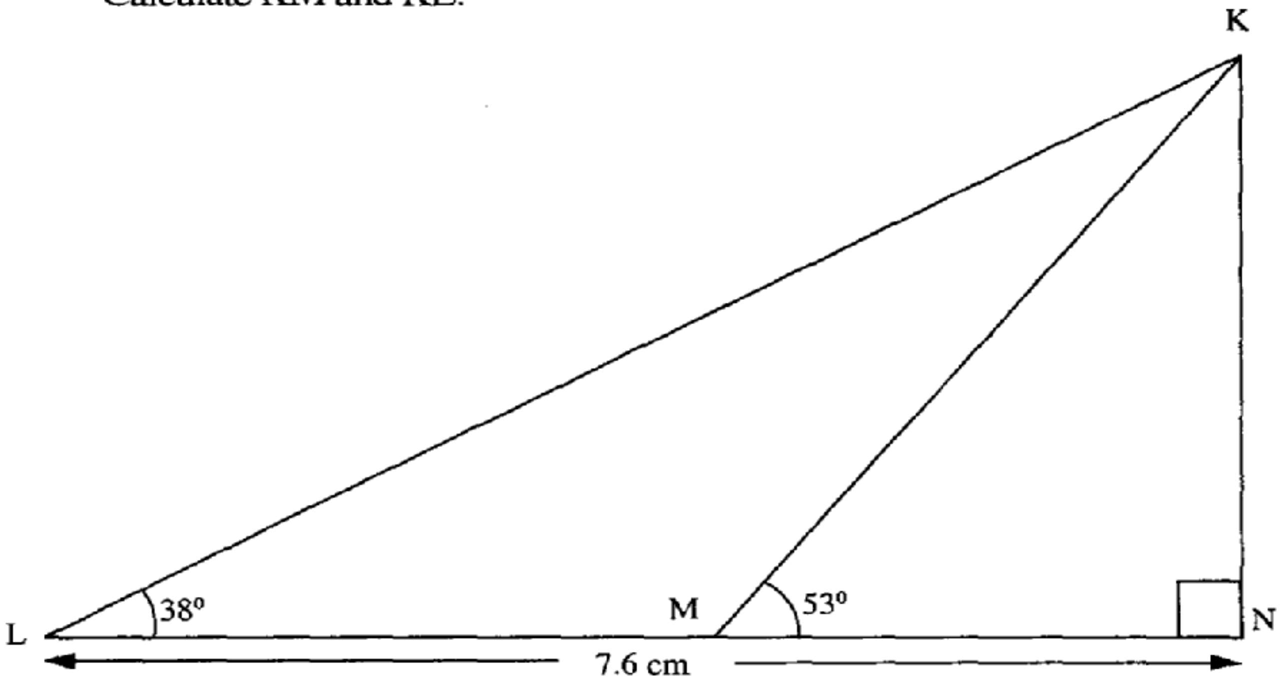


Fig. 9.31

9.5: Sines and Cosines of Complementary Angles

Read from the tables the values of the following:

(i) $\sin 60^0$	(ii) $\cos 20^0$	(iii) $\sin 30^0$
$\cos 30^0$	$\sin 70^0$	$\cos 60^0$

We notice that $\sin 60^0$ is equal to $\cos 30^0$, $\cos 20^0$ is equal to $\sin 70^0$ and $\sin 30^0$ is equal to $\cos 60^0$.

For any two complementary angles x and y, $\sin x = \cos y$ and $\cos x = \sin y$.

Example 9

Find acute angles $\propto$ and β if:

(a) $\sin \propto = \cos 33^0$.

(b) $\cos \beta = \sin 3\beta$

Solution

(a) $\sin \propto = \cos 33$

Therefore, $\propto + 33 = 90$

$\propto = 90^0 - 33^0$

$= 57^0$

(b) $\cos\beta = \sin 3\beta$

Therefore, $\beta + 3\beta = 90^0$

$4\beta = 90^0$

$\beta = 22\frac{1}{2}^0$

Exercise 9.5

1. If A and B are complementary angles and $\sin A = \frac{4}{5}$, find $\cos B$.
2. In figure 9.32, AB = 16 cm and AC = 20 cm. Find:
   (a) $\sin \theta$.
   (b) $\cos \theta$.

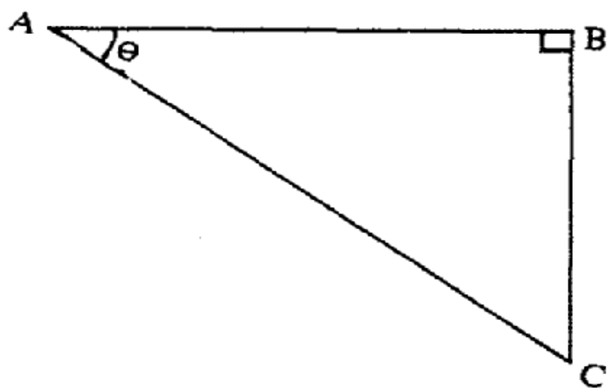


Fig. 9.32

3. X and B are complementary angles. If $\sin B = 0.9975$, find X.
4. A and B are complementary angles. If $A = \frac{1}{2}B$, find:
   (a) $\sin A$.
   (b) $\cos A$.
5. Find the acute angle x, given that $\cos x^0 = \sin 2x^0$.

9.6: Trigonometric Ratios of Special Angles (30^0, 45^0 and 60^0)

Trigonometrical ratios of 30^0, 45^0 and 60^0 can be deduced by the use of an isosceles right-angled triangle and an equilateral triangle as follows.

Tangent, Cosine and Sine of 45^0

Figure 9.33 shows a right-angled isosceles triangle PQR. The dimensions are as shown.

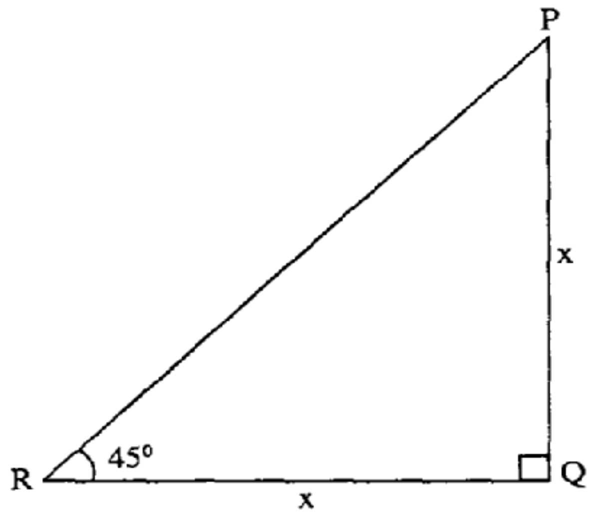


Fig. 9.33

(i) Find the length of PR in terms of x.
(ii) Find cos 45°, sin 45° and tan 45°.

You should have found that:

(iii) $\cos 45^0 = \frac{1}{\sqrt{2}}$ (ii) $\sin 45^0 = \frac{1}{\sqrt{2}}$ (iii) $\tan 45^0 = 1$

This is always true for any positive value of x.

Tangent, Cosine and Sine of 30° and 60°

Figure 9.34 shows an equilateral triangle ABC. AN is the perpendicular bisector of BC. The dimensions are shown.

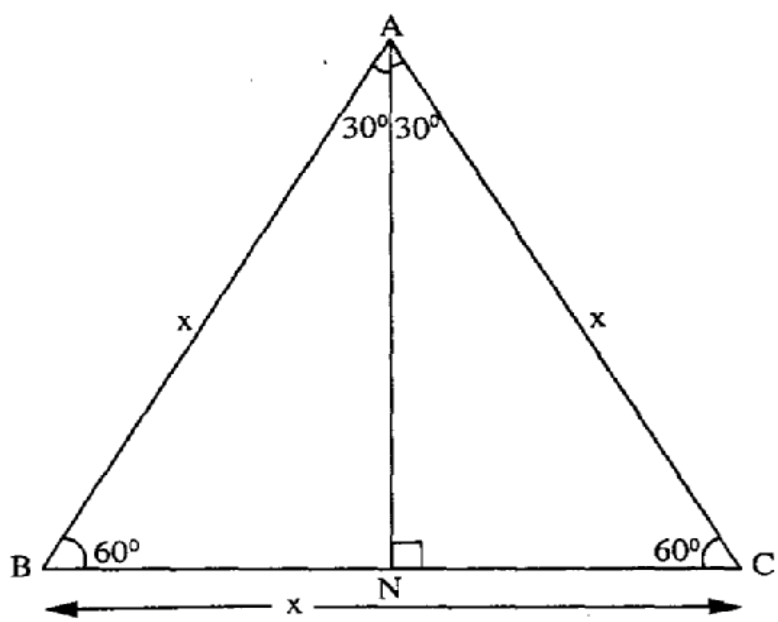


Fig. 9.34

(i) Find an expression for the length of AN in terms of x.
(ii) Find sin 30°, cos 30° and tan 30°.

(iii) Find sin 60^0, cos 60^0 and tan 60^0.

You should have found that:

(i) sin $30^0 = \frac{1}{2}$, cos $30^0 = \frac{\sqrt{3}}{2}$ and tan $30^0 = \frac{1}{\sqrt{3}}$

(ii) sin $60^0 = \frac{\sqrt{3}}{2}$, cos $60^0 = \frac{1}{2}$ and tan $60^0 = \frac{\sqrt{3}}{1} = \sqrt{3}$

Exercise 9.6

1. Simplify the following without using tables (use trigonometric ratios):
   Note: $\sqrt{a} \times \sqrt{a} = a$
   (a) sin 30^0 cos 30^0 (b) 4 cos 45^0 sin 60^0
   (c) 3 cos 30^0 + cos 60^0 (d) tan 45^0 + cos 45^0 sin 45^0
   (e) sin 60^0 cos 30^0 + sin 30^0 cos 60^0
   (f) $\cos^2 60^0 + \sin^2 60^0$ **(sin θ x sin θ is written as $\sin^2\theta$)**
2. An isosceles triangle is such that AB = AC = 8 cm. If the perpendicular distance from A to BC is 6 cm, find:
   (a) the length of BC.
   (b) ∠BAC.
3. The angle made by the arms of an upright pair of dividers and the horizontal is 30^0. The vertical distance from the horizontal to the vertex is 10 cm. Find without using tables:
   (a) the horizontal distance between the tips of the arms.
   (b) the length of the arms.
4. The angle at the vertex of a cone is 90^0. If the slant height is $3\sqrt{2}$ cm, find without using tables:
   (a) the diameter of the cone.
   (b) the height of the cone.
5. Find the height of an equilateral triangle of side x cm. Use the triangle to show that $\sin^2 60^0 + \cos^2 60^0 = 1$ (without using tables).
6. In figure 9.35, ABC is an isosceles triangle. ∠BAC = 120^0 and AB = AC = 13 cm. Calculate BC and the area of the triangle.

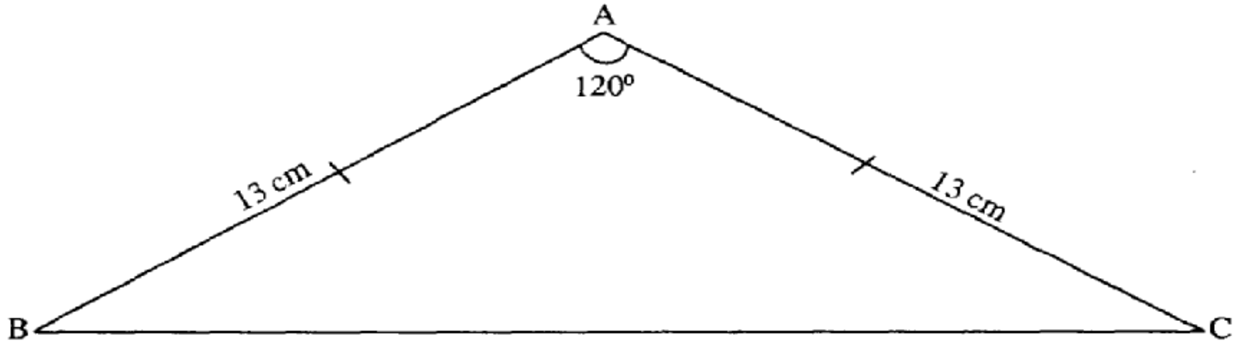


Fig. 9.35

9.7: Logarithms of Tangents, Sines and Cosines

In this section, we extend the use of logarithms in computations involving trigonometry.

For example, to evaluate 234 sin 36^0, we can use the following method:
From tables, sin $36^0 = 0.5878$.
Therefore, 234 sin $36^0 = 234 \times 0.5878$

No	Log
234	2.3692
0.5878	$\bar{1}.7692$ +
1.375×10^2	2.1384

Therefore , 234 sin 36 = 137.5
Alternatively, log sin 36° can be read directly from the tables of logarithms of sines.

No	Log
234	2.3692
sin 36^0	$\bar{1}.7692$ +
1.375×10^2	2.1384

Therefore, 234 sin 36^0 = 137.5
Similarly, values of log(cos x) and log(tan x) can be read from their respective tables.

Example 10

Evaluate $\dfrac{69.6 \cos 42}{\sin 64}$

Solution

No	Log
69.6	1.8426
cos 42^0	$\bar{1}.8711$ +
	1.7137
sin 64^0	$\bar{1}.9537$ –
5.75×10^1	1.7600

Therefore, $\dfrac{69.6 \cos 42^0}{\sin 64^0} = 57.54$

Example 11

Three towns P, Q and R are such that Q is 150 km from P on a bearing of 043^0 (see figure 9.36). The bearing of R from P is 133^0 and the bearing of R from Q is 160^0. Calculate the distance of R from P, Q from R and the bearing of P from R.

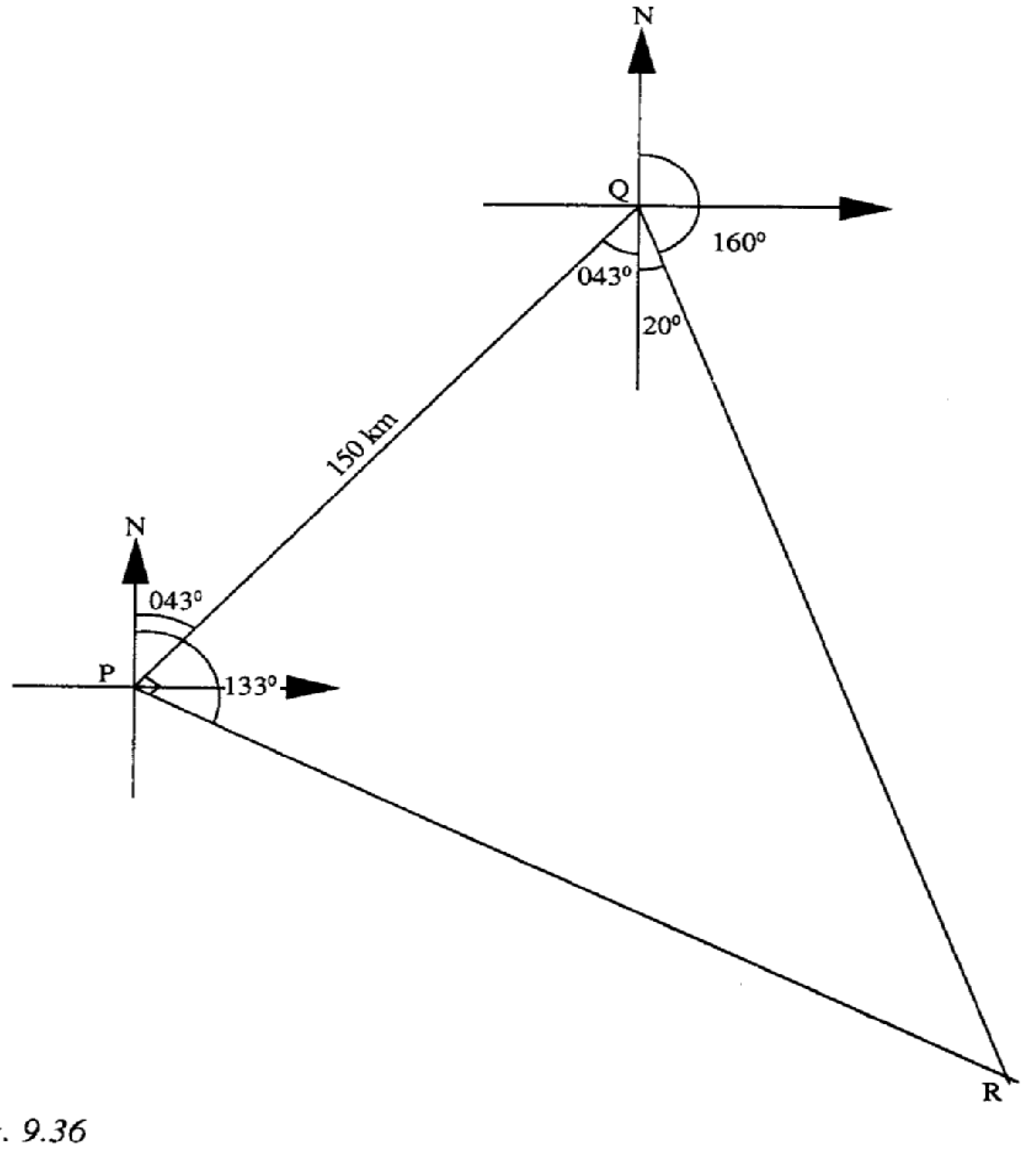


Fig. 9.36

Solution

In the figure, ΔPQR is right-angled at P and $\angle PQR = 63^0$.

Hence, $\tan 63^0 = \dfrac{PR}{150}$

$150 \tan 63^0 = PR$

No	Log
150	2.1761
tan 63^0	0.2928 +
2.944×10^2	2.4689

Therefore, PR = 294.4 km

In the right-angled ΔPQR, QR is the hypotenuse;

$$\cos 63^0 = \frac{150}{QR}$$

$$QR = \frac{150}{\cos 63^0}$$

No	Log
150	2.1761
cos 63^0	$\bar{1}.6570$ –
3.305×10^2	2.5191

Therefore, QR = 330.5 km
The bearing of P from R;

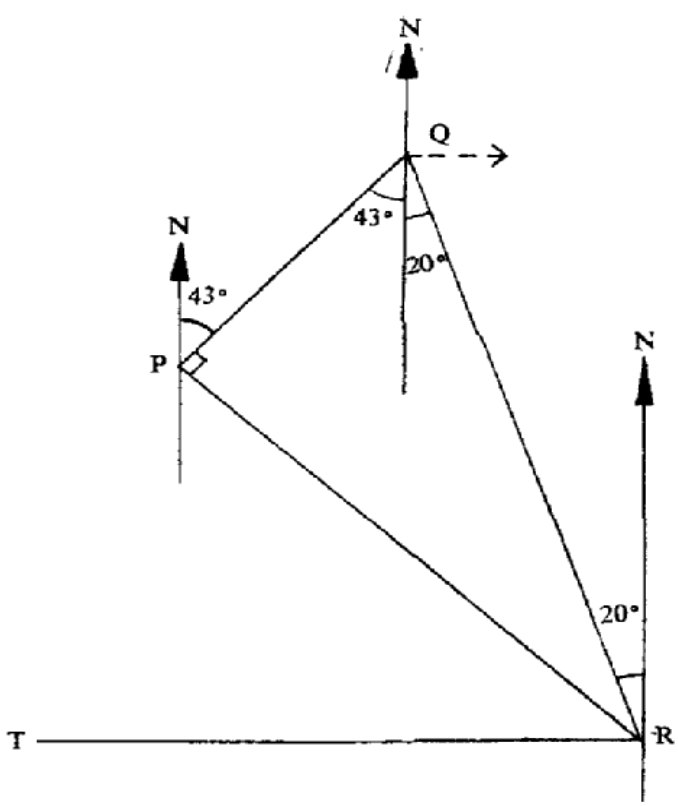


Fig. 9.37

From figure 9.37,

$\angle PRQ = 180^0 - (90 + 63)^0$
$= 27^0$

$\angle PRT = 90^0 - (20 + 27)^0$
$= 43^0$

Therefore, the bearing of P from R = $270^0 + 43^0 = 313^0$

Exercise 9.7

1. Evaluate:
   (a) $8.52 \tan 42.2^0$ (b) $7.9 \sin 79^0$
   (c) $\dfrac{69}{\cos 63.6^0}$ (d) $\dfrac{7\cos 50.2^0}{9.5 \sin 60^0}$
2. Use figure 9.38 to find the unknown angles and sides in each case:
   (a) $\angle PRQ = 46^0$, $q = 7.83$ (b) $p = 13.6$, $q = 17.2$
   (c) $\angle QPR = 56^0 10'$, $p = 4.53$ (d) $\angle QPR = 47^0 35'$, $r = 3.47$

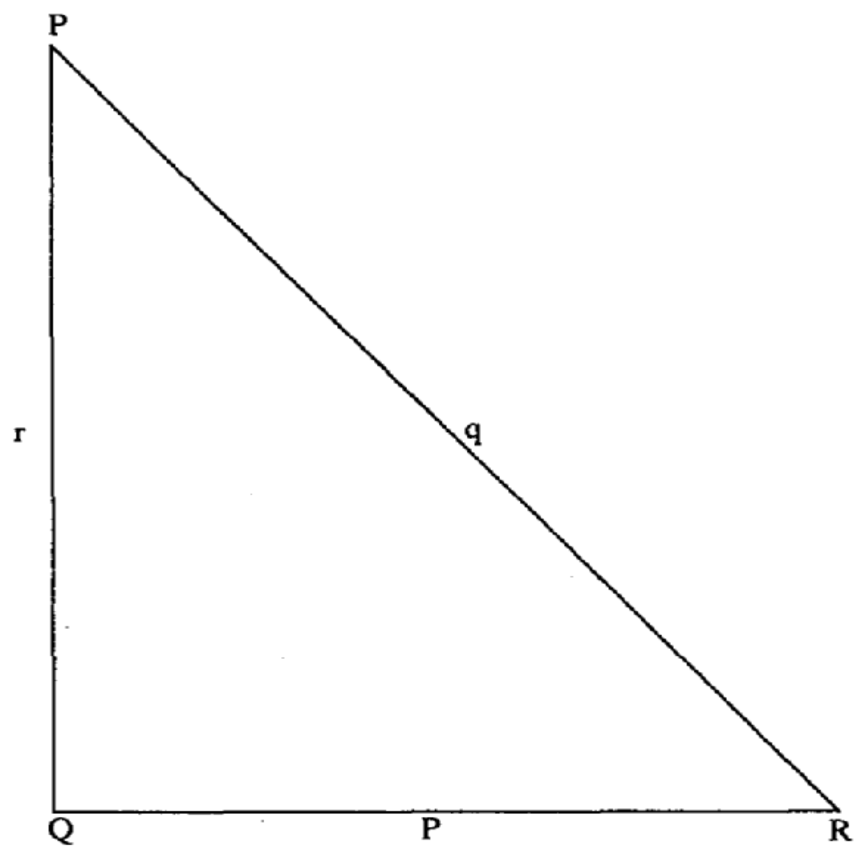


Fig. 9.38

3. In a triangle PQR, $\angle PQR = \angle PRQ = 58^0$ and QR = 5.2 cm. Calculate the length of PQ.
4. In a triangle PQR, QR = 5.2 cm and PQ = PR = 8.2 cm. Calculate:
   (a) $\angle PQR$ and $\angle QPR$.
   (b) the area of ΔPQR.
5. In a triangle XYZ, XY = XZ = 4.1 cm. XN is the altitude of the triangle, which is 3.2 cm long. Calculate:
   (a) the base angles of the triangle.
   (b) the size of the vertex angle.
   (c) area of triangle XYZ.
6. Calculate the area of a parallelogram PQRS in which PS = 3.1 cm, PQ = 7.2 cm and $\angle SPQ = 73^0$.
7. Figure 9.39 shows a regular pentagon PQRST of side 5.2 cm. Find the length of PM if $\angle PMS = 90^0$ and M bisects RS.

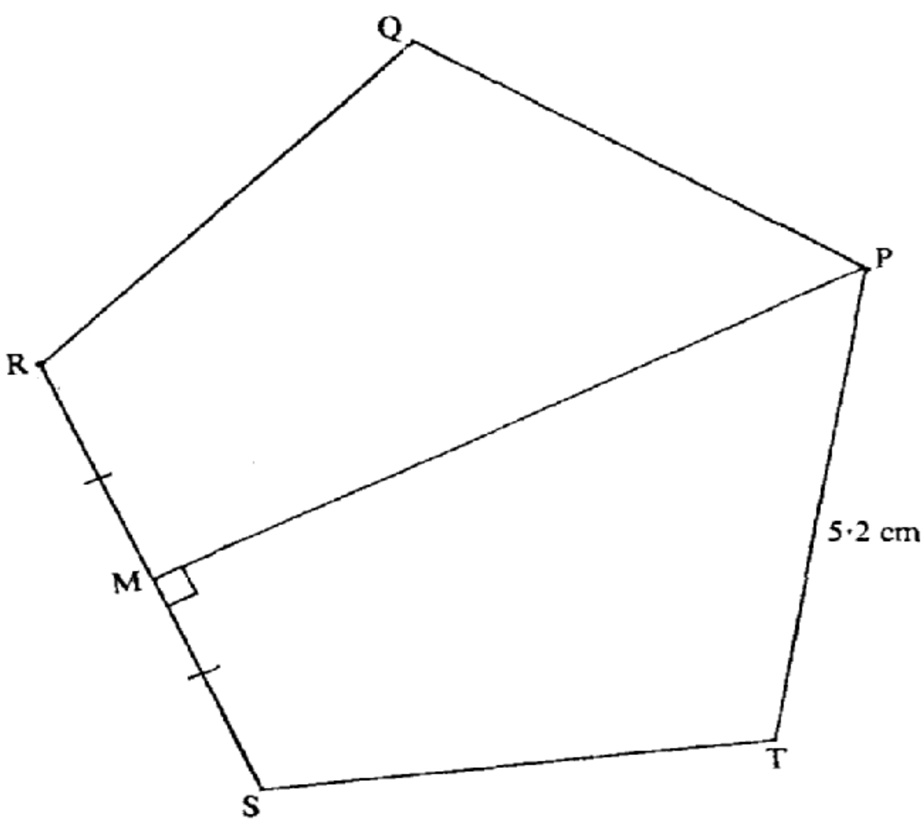


Fig. 9.39

8. In figure 9.40, PQ = 2 500 m, UT = 1 000 m and TS = 2 350 m. PQR is a straight line parallel to UT and PU is parallel to TS. Calculate PR to the nearest kilometre.

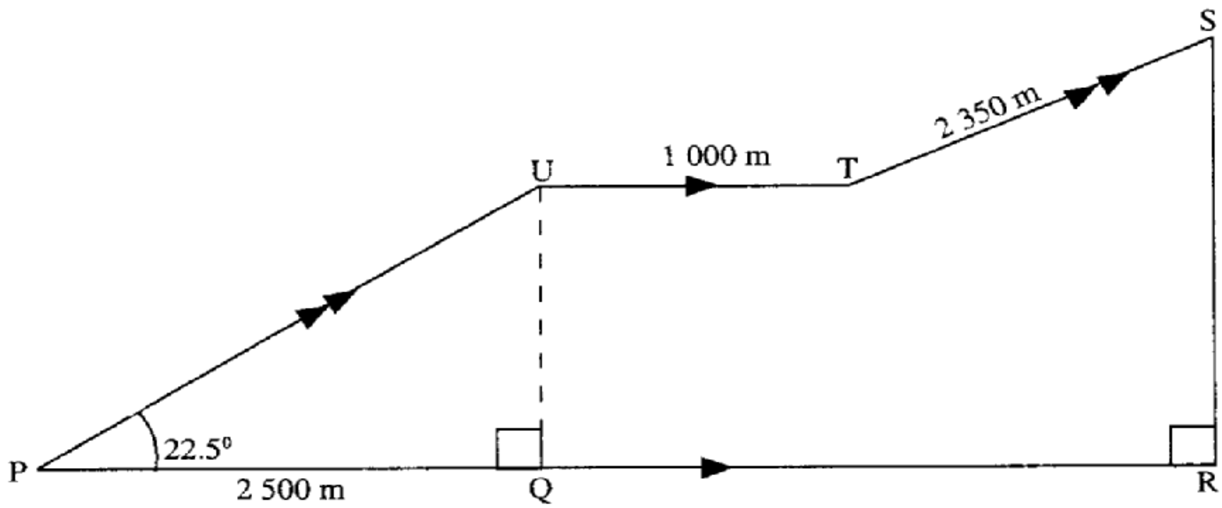


Fig. 9.40

9. A ship moves 4.5 km on a bearing of 330°. It then changes its course to a bearing of 270° and moves 11 km, then changes its course again to a bearing of 315° for 8 km. Find, to the nearest km, how far north and west the ship is from the starting point.

Chapter Ten

AREA OF A TRIANGLE

10.1: Area of a Triangle, given Two Sides and an Included Angle

The area of a triangle is usually given by the formula $A = \frac{1}{2}bh$, where b is the base and h the height. At times, situations arise which make it difficult to apply this formula directly, as in the following examples.

Example 1

Find the area of triangle ABC in which AB = 12 cm, BC = 16 cm and $\angle ABC = 30^0$.

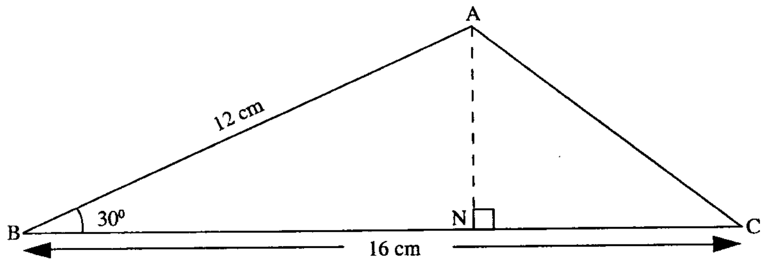


Fig. 10.1

Solution

From A, drop a perpendicular to meet BC at N. AN is the height of triangle ABC.

$$\text{In triangle ABN, } \sin 30^0 = \frac{AN}{AB}$$

$$= \frac{AN}{12}$$

$$\text{Therefore, the height of } \Delta ABC = 12 \sin 30^0$$

$$= 12 \times \tfrac{1}{2}$$

$$= 6 \text{ cm}$$

$$\text{Area of triangle ABC} = \tfrac{1}{2} \times BC \times AN$$

$$= \tfrac{1}{2} \times 16 \times 6$$

$$= 48 \text{ cm}^2$$

Generally, if the lengths of the two sides and an included angle of a triangle are given (see figure 10.2), then the area A of the triangle is given by;

$A = \frac{1}{2}ab\sin\theta$.

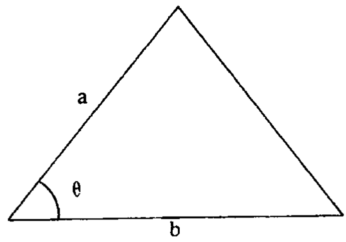


Fig. 10.2

Example 2

Figure 10.3 shows a triangle ABC in which AB = 6 cm, BC = 7 cm and $\angle ABC = 50^0$. Find the area of the triangle.

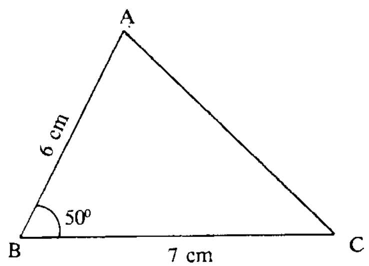


Fig. 10.3

Solution

Using the formula $A = \frac{1}{2}ab\sin\theta$, a = 6 cm, b = 7 cm and $\theta = 50^0$.

$$\begin{aligned}\text{Therefore, area } &= \tfrac{1}{2} \times 6 \times 7 \ \sin 50^0 \\ &= 3 \times 7 \times 0.7660 \\ &= 16.09 \text{ cm}^2\end{aligned}$$

10.2: Area of Triangle, given the Three Sides

Example 3

Find the area of a triangle ABC in which AB = 5 cm, BC = 6 cm and AC = 7 cm.

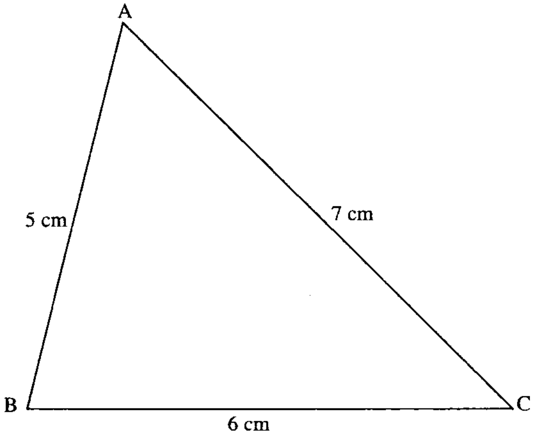


Fig. 10.4

When only the three sides of a triangle are given, either use scale drawing and construct a perpendicular height to one of the sides, or use the formula:

$A = \sqrt{s(s-a)(s-b)(s-c)}$ (Hero's formula),

where $s = \frac{1}{2}$ of the perimeter of the triangle

$= \frac{1}{2}(a + b + c)$

a, b and c are the lengths of the sides of the triangle.
In the figure, $a = 6$, $b = 7$ and $c = 5$. Therefore;

$s = \frac{1}{2}(6 + 7 + 5)$
$= 9$

and $A = \sqrt{9(9-6)(9-7)(9-5)}$

$= \sqrt{9 \times 3 \times 2 \times 4}$

$= \sqrt{216}$

$= 14.70 \text{ cm}^2$

Exercise 10.1

1. Find the area of each of the triangles below:

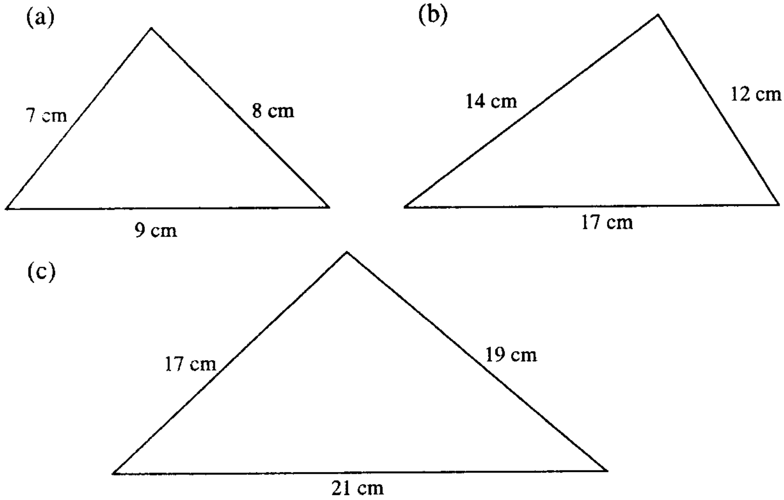


Fig. 10.5

2. Find the area of each of the following figures:

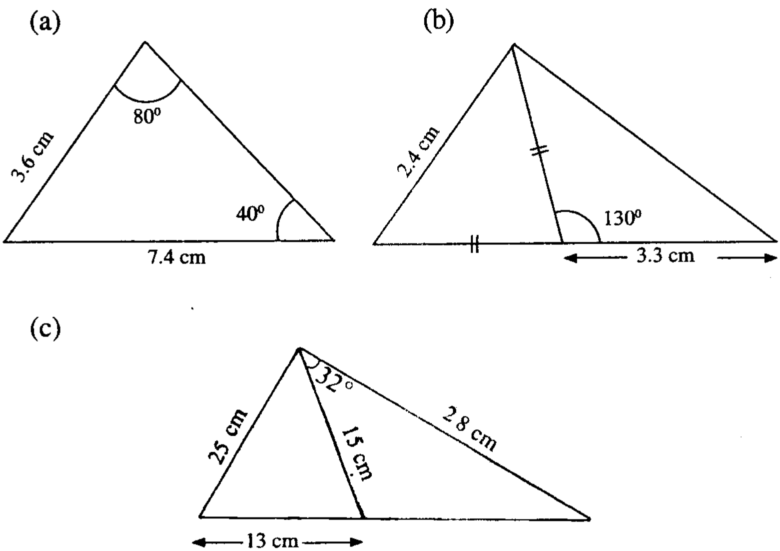


Fig. 10.6

3. A triangle has sides 10 cm, 7 cm and 9cm. Find:
   (a) its area.
   (b) the sizes of its angles.
4. The area of triangle ABC in figure 10.7 is 28.1 cm². Find:
   (a) the length of the perpendicular from A to BC.
   (b) the length of BC.

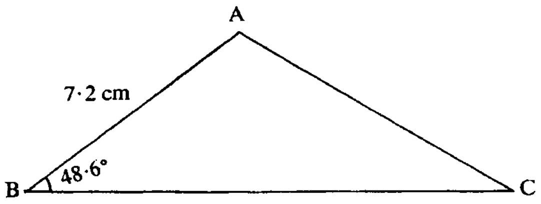


Fig. 10.7

5. In a triangle XYZ, XY = 7 cm, YZ = 24 cm and XZ = 25 cm. Find the length of the perpendicular from vertex Y to side XZ.
6. The perimeter of a triangle is 22 cm. If one of the sides is 9 cm, find the other sides if the area of the triangle is 20.976 cm².
7. A traditional stool has a triangular top which measures 27 cm, 35 cm and 42 cm. Calculate the area of the top.
8. A triangular flower garden measures 10 m, 15 m and 24 m. Find the area of the garden.

Chapter Eleven

AREA OF QUADRILATERALS

11.1: Quadrilaterals

A quadrilateral is a four-sided figure. Some examples of quadrilaterals are rectangle, square, rhombus, parallelogram, trapezium and kite.

Revision

Area of a rectangle, $A = lb$.

Example 1

Find the area of a rectangle whose length is 12 cm and width 7 cm.

Solution

$A = lb$

$= 12 \times 7\ cm^2$

$= 84\ cm^2$

11.2: Area of Parallelogram

A parallelogram is a quadrilateral whose opposite sides are equal and parallel. Area of parallelogram is given by;

$A = bh$, where b is the base and h the perpendicular distance between the given pair of parallel sides, i.e., the height. For example, the area of a parallelogram whose base is 8 cm and height 6 cm is;

$A = bh$

$= 8 \times 6\ cm^2$

$= 48\ cm^2$

Example 2

ABCD is a parallelogram of sides 5 cm and 10 cm. If $\angle ABC$ is 70^0, find the area of the parallelogram.

Solution

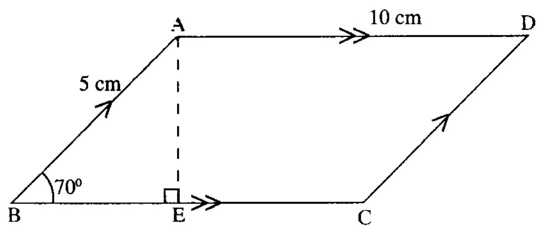


Fig. 11.1

From A, drop a perpendicular to meet BC at E. Considering ΔABE;
$AE = 5 \sin 70^\circ$

$$\begin{aligned}\text{Area of ABCD} &= BC \times AE \\ &= 10 \times 5 \sin 70^\circ \\ &= 50 \sin 70^\circ \\ &= 50 \times 0.9397 \\ &= 46.99 \text{ cm}^2\end{aligned}$$

11.3: Area of a Rhombus

A rhombus is a special case of a parallelogram. All its sides are equal and the diagonals bisect at 90^0.

Example 3

Find the area of figure 11.2 below, given that PR = 4 cm, QS = 7 cm, PQ = QR = RS = SP and QR//PS.

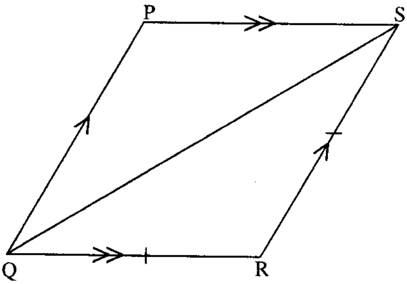


Fig 11.2

Solution

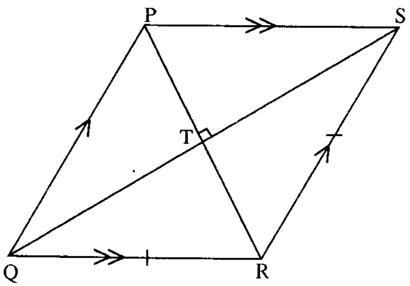


Fig. 11.3

Figure 11.3 is a rhombus. The diagonals of a rhombus bisect at 90°.

$QT = \frac{1}{2} \times 7 \text{ cm}$

$= 3.5 \text{ cm}$

Area of $\Delta PQR = \frac{1}{2} \times 4 \times 3.5 \text{ cm}^2$

$= 7 \text{ cm}^2$

Area of PQRS $= 2 \times$ area of ΔPQR

$= 2 \times 7 \text{ cm}^2$

$= 14 \text{ cm}^2$

11.4: Area of a Trapezium

A trapezium is a quadrilateral with only two of its opposite sides being parallel.

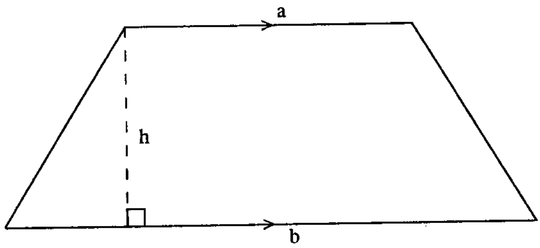


Fig. 11.4

The area of a trapezium is given by the formula;

$A = \left(\frac{a + b}{2}\right)h$, where a and b are the two parallel sides and h the perpendicular distance between them.

Example 4

Figure 11.4 is a trapezium in which PS//QR, PS = 15 cm, QR = 20 cm, RS = 8 cm and $\angle QRS = 35^0$. Calculate the area of the trapezium.

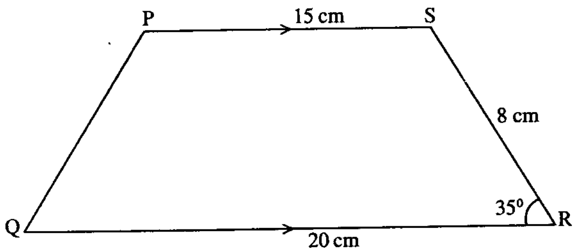


Fig. 11.5

Solution

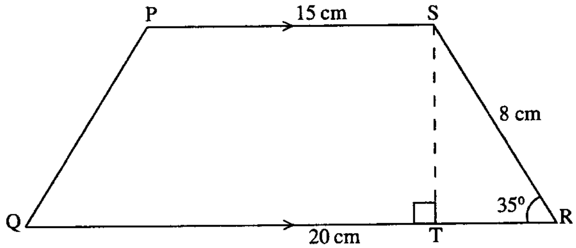


Fig. 11.6

From S, drop a perpendicular to meet QR at T. ST is the height of the trapezium.

$$\text{Area of PQRS} = \left(\frac{PS + QR}{2}\right) \times ST$$

$$= \left(\frac{15 + 29}{2}\right) \times 8 \sin 35^0$$

$$= 140 \sin 35^0$$

No	Log
140	2.1461
sin 35°	$\bar{1}.7586$ +
80.30	1.9047

$\therefore$ Area of PQRS = 80.3 cm^2

11.5: Area of a Kite

Figure 11.7 represents a kite.

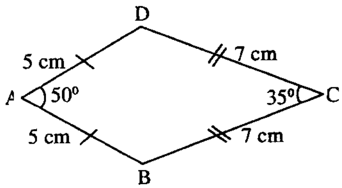


Fig. 11.7

Example 5

Find the area of figure 11.7 above.

Solution

Join D to B.

$$\text{Area of } \Delta ABD = \tfrac{1}{2} \times 5 \times 5 \sin 50^0$$

$$= 12.5 \sin 50^0$$

No	Log
12.5	1.0969
sin 50^0	$\bar{1}.8843$ +
9.576	0.9812

Area of ΔABD = 9.576 cm^2

Area of ΔBCD = $\frac{1}{2} \times 7 \times 7 \sin 35^0$

= $24.5 \sin 35^0$

No	Log
24.5	1.3892
sin 35^0	$\bar{1}.7586$ +
14.05	1.1478

Area of ΔBCD = 14.05 cm^2

Area of ABCD = area of ΔABD + area of ΔBCD

= 9.576 + 14.05

= 23.63 cm^2

11.6: Area of Regular Polygons

Any regular polygon can be divided into isosceles triangles by joining the vertices to the centre. The number of isosceles triangles formed is equal to the number of sides of the polygon.

Example 6

Figure 11.8 represents a regular octagon ABCDEFGH with O as its centre. If OA is 6 cm, find its area.

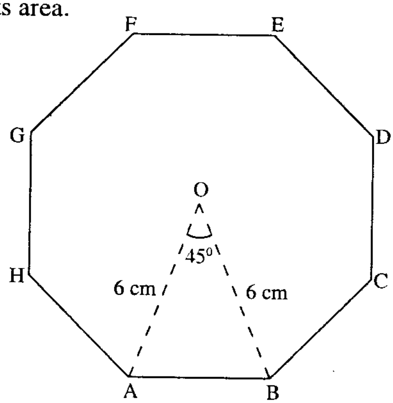


Fig 11.8

Solution

Area of triangle AOB $= \frac{1}{2} \times 6 \times 6 \sin 45^\circ$

$= \frac{1}{2} \times 36 \times 0.7071\ cm^2$

There are 8 triangles since an octagon has 8 sides.

Therefore, total area $= 8 \text{ x } (\frac{1}{2} \times 36 \times 0.7071)$

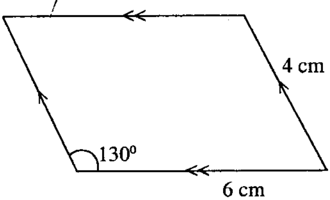
$= 8 \times 18 \times 0.7071$

$= 101.8\ cm^2$

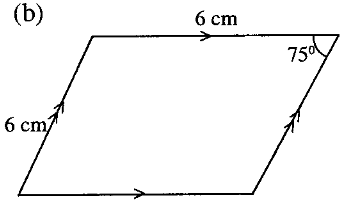
Exercise 11.1

1. Find the area of each of the following figures:

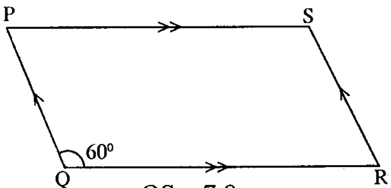
(a)



(b)



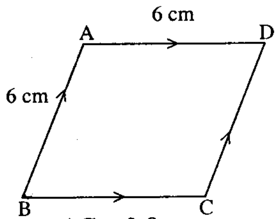
(c)



QS = 7.8 cm

PR = 5 cm

(d)



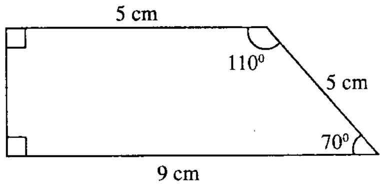
AC = 3.8 cm

BD = 6 cm

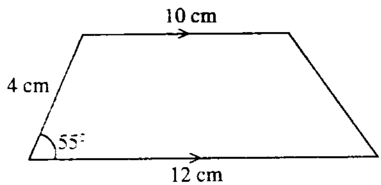
Fig. 11.9

2. Calculate the area of each of the figures below:

(a)



(b)



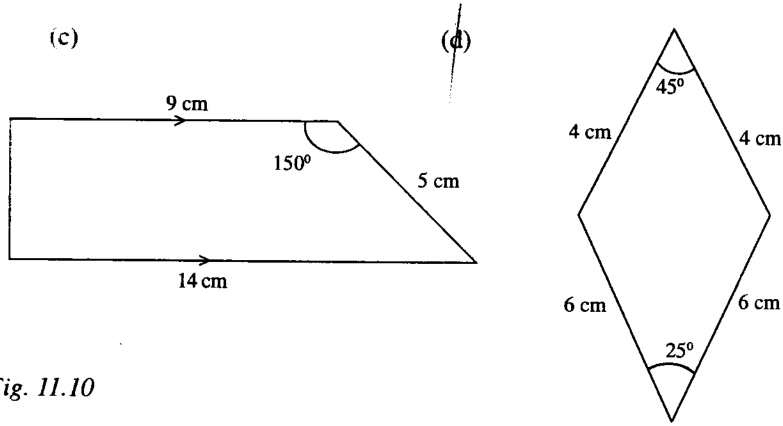


Fig. 11.10

3. Find the area of a regular hexagon of side 4.8 cm.
4. Find the area of a rhombus whose diagonals are 9.6 cm and 6.0 cm long.
5. A pentagon PQRST is such PQ = 2 cm, QR = 3 cm, RS = 1.5 cm and PT = 4 cm. If $\angle TPQ = \angle QRS = 90^0$ and $\angle TQS = 36^0$, find the area of the pentagon.
6. Find the length of the sides of a regular heptagon of area 168 cm^2.
7. The area of a parallelogram is 121 cm^2 and its sides are 11 cm and 16 cm long. Find the sizes of all the angles of the parallelogram.
8. One of the angles of a rhombus is 120^0 and its sides are 15 cm long. Find the area of the rhombus.
9. ABCD is a parallelogram of area 120 cm^2. Its base is 10 cm and $\angle ABC = 30^0$. Find the length of the other side of the parallelogram.
10. Find the value of a in figure 11.11, if its area is 128 cm^2.

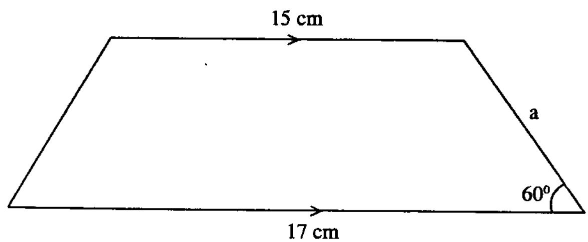


Fig. 11.11

Chapter Twelve

AREA OF PART OF A CIRCLE

12.1: Sector

A sector is a region bounded by two radii and an arc. A minor sector is one whose area is less than a half of the area of the circle while a major sector is one whose area is greater than a half of the area of the circle, as in figure 12.1.

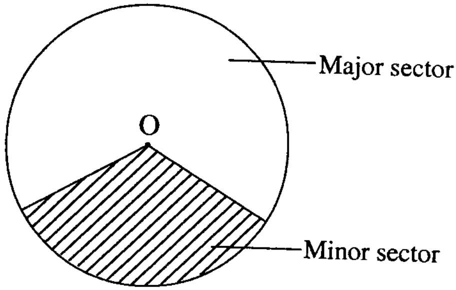


Fig. 12.1

The Area of a Sector

Suppose we want to find the area of the shaded region in figure 12.2.

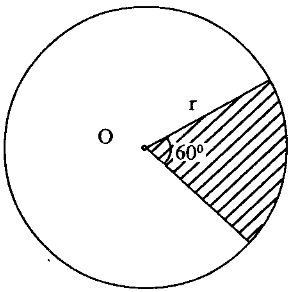


Fig. 12.2

The area of the whole circle is πr^2. The whole circle subtends 360^0 at the centre. Therefore, 360^0 corresponds to πr^2.

1^0 corresponds to $\dfrac{1}{360} \times \pi r^2$

60^0 corresponds to $\dfrac{60}{360} \times \pi r^2$

In general, the area of a sector subtending an angle θ at the centre of the circle is given by; $A = \frac{\theta}{360} \times \pi r^2$

Example 1

Find the area of the sector of a circle of radius 3 cm if the angle subtended at the centre is 140^0. (Take $\pi = \frac{22}{7}$)

Solution

Area A of a sector is given by;

$$A = \frac{\theta}{360} \times \pi r^2$$

$$\therefore \text{Area} = \frac{140}{360} \times \frac{22}{7} \times 3^2$$

$$= 11 \text{ cm}^2$$

Example 2

The area of a sector of a circle is 38.5 cm^2. Find the radius of the circle if the angle subtended at the centre is 90^0. (Take $\pi = \frac{22}{7}$)

Solution

From $A = \frac{\theta}{360} \pi r^2$, we get;

$$\frac{90}{360} \times \frac{22}{7} \times r^2 = 38.5$$

$$\therefore r^2 = \frac{38.5 \times 360 \times 7}{90 \times 22}$$

$$r^2 = 49$$

Thus, $r = 7$ cm

Example 3

The area of a sector of a circle radius 63 cm is $4\,158 \text{ cm}^2$. Calculate the angle subtended at the centre of the circle. (Take $\pi = \frac{22}{7}$)

Solution

$$4\,158 = \frac{\theta}{360} \times \frac{22}{7} \times 63 \times 63$$

$$\theta = \frac{4\ 158 \times 360 \times 7}{22 \times 63 \times 63}$$

$$= 120^0$$

Exercise 12.1

1. A sector of a circle of radius r subtends an angle θ. Calculate the area of the sector if:
   (a) $r = 1.4$ cm, $\theta = 30^0$ (b) $r = 2.1$ cm, $\theta = 45^0$
   (c) $r = 8$ cm, $\theta = 33^0$ (d) $r = 8.4$ cm, $\theta = 60^0$
   (e) $r = 9.1$ cm, $\theta = 24^0$ (f) $r = 4$ cm, $\theta = 259^0$
   (g) $r = 10$ cm, $\theta = 301^0$
2. A floodlight can spread its illumination over an angle of 50^0 to a distance of 49 cm. Calculate the area that is floodlit.
3. The shaded region in figure 12.3 shows the area swept out on a flat windscreen by a wiper. Calculate the area of this region.

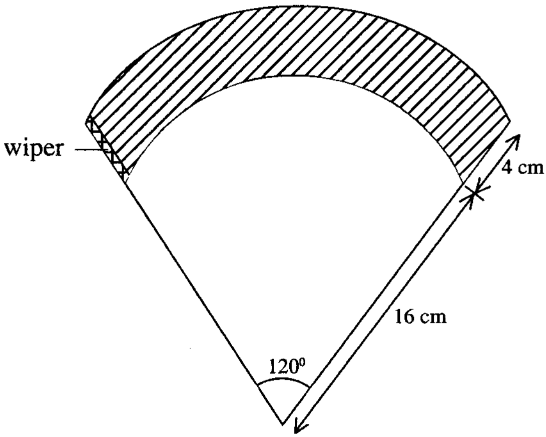


Fig. 12.3

4. The two arms of a pair of dividers are spread so that the angle between them is 45^0. Find the area of the sector formed if the length of an arm is 8.4 cm. (Take $\pi = \frac{22}{7}$)
5. A goat is tethered at the corner of a fenced rectangular grazing field. If the length of the rope is 21 m, what is the grazing area?

12.2: Area of a Segment of a Circle

A segment is a region of a circle bounded by a chord and an arc. In figure 12.4, the shaded region is a segment of the circle with centre O and radius r. AB = b cm, OM = h cm and $\angle AOB = \theta^0$.

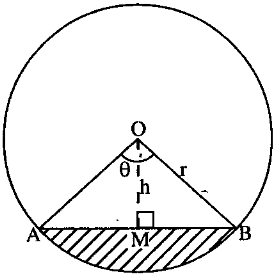


Fig. 12.4

The area of the segment = area of minor sector OAB – area of triangle OAB

$$= \frac{\theta}{360} \times \pi r^2 - \tfrac{1}{2}bh$$

Example 4

Figure 12.5 is a circle with centre O and radius 5 cm. If ON = 3 cm, AB = 8 cm and $\angle AOB = 106.3°$, find the area of the shaded region:

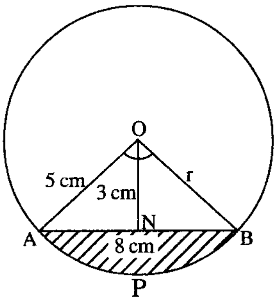


Fig. 12.5

Solution

Area of segment = area of sector OAPB – area of triangle OAB

$$= \left[\frac{106.3}{360} \times 3.142 \times 5^2\right] - \tfrac{1}{2} \times 8 \times 3$$

$$= \frac{106.3}{360} \times 25 \times 3.142 - 12$$

$$= 23.19 - 12$$

$$= 11.19 \text{ cm}^2$$

Exercise 12.2

1. A chord XY of length 12 cm is drawn in a circle with centre O and radius 10 cm, as in figure 12.6.

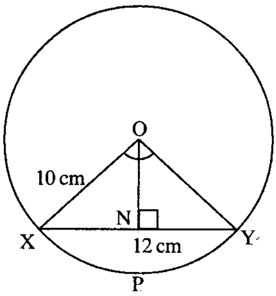


Fig. 12.6

Calculate:
(a) the distance ON.
(b) the area of the sector OXPY.
(c) the area of triangle OXY.
(d) the area of the minor segment.
(e) the area of the major segment.

2. A chord XY subtends an angle of 120° at the centre of a circle of radius 13 cm. Calculate the area of the minor segment.

3. Figure 12.7 shows a circle with centre O and radius $4\sqrt{2}$ cm. If the length of the chord AB is 8 cm, show that the shaded area is $(8\pi - 16)$ cm^2.

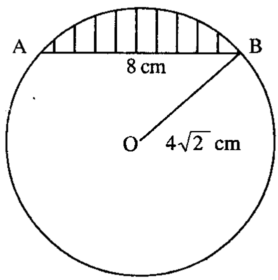


Fig. 12.7

4. In figure 12.8, ADC is a chord of a circle with centre O passing through A, B and C. BD is a perpendicular bisector of AC . AD = 3 cm and BD = 1 cm.

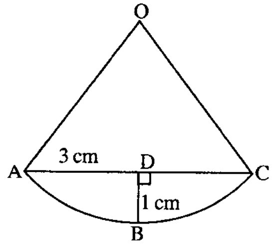


Fig. 12.8

Calculate:
(a) the radius OA of the circle.
(b) the area of the sector OABC.
(c) the area of the segment ABCD.

5. Figure 12.9 shows an arc ACE of a circle with centre O and radius 6 cm. If BC = CD = 4 cm, calculate the area of the shaded region.

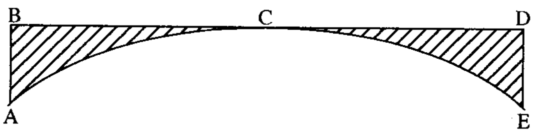


Fig. 12.9

6. In figure 12.10, ABC is an arc of a circle with centre O and radius 7 cm. The arc subtends an angle of 60° at the centre and AE = DC = AC = ED = 7 cm. Calculate the area of the figure ABCDE.

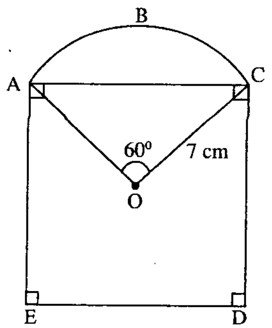


Fig. 12.10

12.3: Area of Common Region between Two Intersecting Circles

Example 5

Figure 12.11 shows two circles of radii 8 cm and 6 cm with centres O_1 and O_2 respectively. The circles intersect at points A and B. The lines O_1O_2 and AB are perpendicular to each other. If the common chord AB is 9 cm, calculate the area of the shaded region.

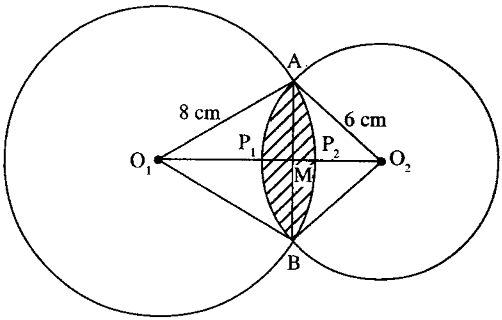


Fig. 12.11

Solution

From ΔAO_1M;

$$O_1M = \sqrt{8^2 - 4.5^2} = \sqrt{43.75} = 6.614 \text{ cm}$$

From ΔAO_2M

$$O_2M = \sqrt{6^2 - 4.5^2} = \sqrt{15.75} = 3.969 \text{ cm}$$

The area between the intersecting circles is the sum of the areas of segments AP_1B and AP_2B. Area of segment AP_1B = area of sector O_2AP_1B – area of ΔO_2AB

Using trigonometry, $\sin \angle AO_2M = \dfrac{AM}{AO_2}$

$$= \frac{4.5}{6}$$

$$= 0.75$$

$\angle AO_2M = 48.59^0$
$\angle AO_2B = 2\angle AO_2M$
$= 2 \times 48.59$
$= 97.18^0$

Area of segment $AP_1B = \frac{97.18}{360} \times 3.142 \times 6^2 - \frac{1}{2} \times 9 \times 3.969$
$= 30.53 - 17.86$
$= 12.67 \text{ cm}^2$

Area of segment AP_2B = area of sector O_1AP_2B – area of ΔO_1AB

Using trigonometry, $\sin \angle AO_1M = \frac{AM}{AO_1}$
$= \frac{4.5}{8}$
$= 0.5625$

$\angle AO_1M = 34.23^0$
$\angle AO_1B = 2\angle AO_1M$
$= 2 \times 34.23^0$
$= 68.46^0$

Area of segment $AP_2B = \frac{68.46}{360} \times 3.142 \times 8^2 - \frac{1}{2} \times 9 \times 6.614$
$= 38.24 - 29.76$
$= 8.48 \text{ cm}^2$

Therefore, the area of the region between the intersecting circles is given by;
area of segment AP_1B + area of segment $AP_2B = 12.67 + 8.48$
$= 21.15 \text{ cm}^2$

Exercise 12.3

1. Figure 12.12 shows two intersecting circles of radius 9 cm each, with centres O_1 and O_2.

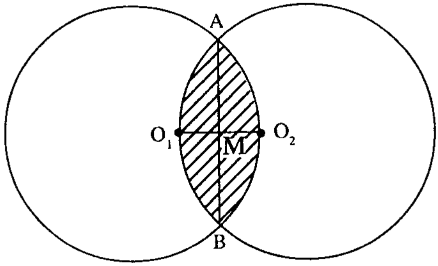


Fig. 12.12

Find:

(a) the length of the common chord AB.

(b) the area common to the two circles.

2. Figure 12.13 shows two intersecting circles with centres O_1 and O_2 having radii 4 cm and 3 cm respectively. $\angle AO_1B = 57.64^0$ and $\angle AO_2B = 80^0$. O_1O_2 is a perpendicular bisector of AB.

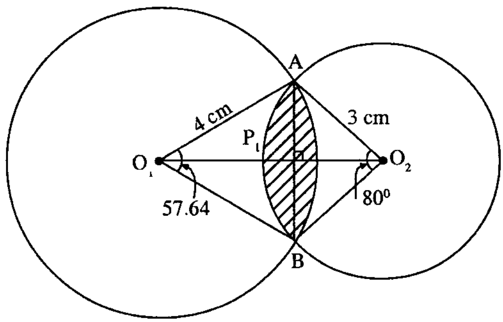


Fig. 12.13

Calculate :

(a) the length of AB.

(b) the area of the shaded region.

3. Find the area of the shaded region in figure 12.14, given that the two circles with centres O_1 and O_2 have radii 10 cm and 8 cm respectively, $\angle AO_1B = 90^0$ and $\angle AO_2B = 124.2^0$.

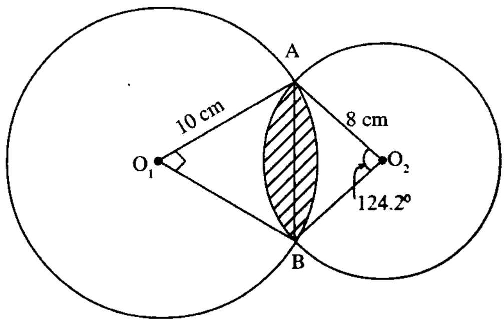


Fig. 12.14

4. Find the common area between the two intersecting circles in figure 12.15. The circles with centres O_1 and O_2 have radii 18 cm and 12 cm respectively and the chord AB is 18 cm long.

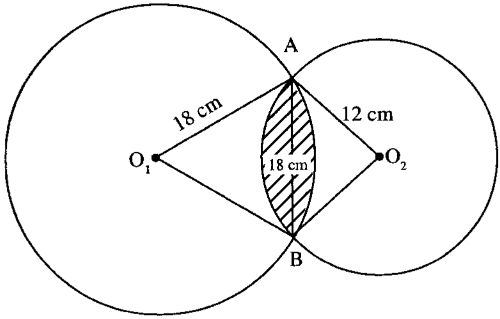


Fig. 12.15

Chapter Thirteen

SURFACE AREA OF SOLIDS

13.1: Introduction

Solids are objects with definite shape and size. A solid can be regular or irregular. In this topic, we are going to consider regular solids.

13.2: Surface Area of a Prism

A prism is a solid with uniform cross-section. Figure 13.1 shows prisms with different cross-sections.

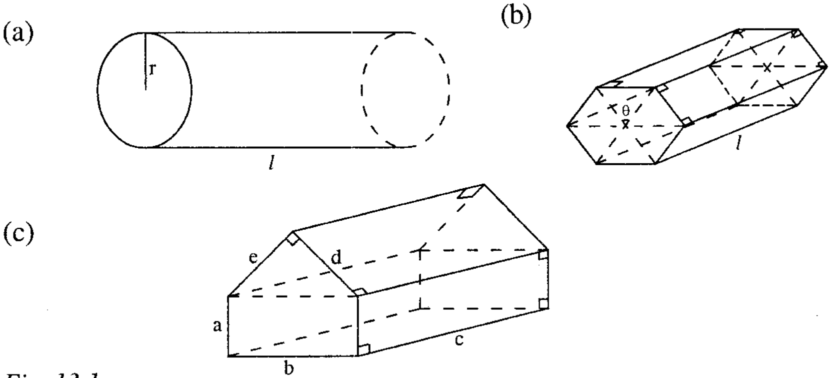


Fig. 13.1

The surface area of a prism is the sum of the areas of its faces.

Example 1

Find the surface area of each of the prisms in figure 13.1 if:

(a) $r = 2.8$ cm, $l = 13$ cm.

(b) $r = 6$ cm, $\theta = 60^0$, $l = 15$ cm.

(c) $a = 4$ cm, $b = 5$ cm, $c = 8$ cm, $d = 4$ cm and $e = 3$ cm.

Solution

(a) Surface area of a cylinder = surface area of two circular ends + area of the curved surfaces

$$= 2\pi r^2 + 2\pi r l$$

$$= 2(\tfrac{22}{7} \times 2.8 \times 2.8) + (2 \times \tfrac{22}{7} \times 2.8 \times 13)$$

$$= 49.28 \text{ cm}^2 + 228.8 \text{ cm}^2$$

$$= 278.08 \text{ cm}^2$$

(b) Surface area $=$ area of hexagonal ends $+$ area of six rectangular faces

$$= [6(\tfrac{1}{2}ab\sin\theta)] \times 2 + 6\,(l \times w)$$

$$= 2[6(\tfrac{1}{2} \times 6 \times 6 \times \sin 60^0)] + 6\,(6 \times 15)$$

$$= 2[6(\tfrac{1}{2} \times 36 \times 0.8660)] + 6 \text{ x } 90$$

$$= (187.06 + 540)\text{ cm}^2$$

$$= 727.06\text{ cm}^2$$

(c) Surface area = area of the ends + area of the five rectangular faces

$$= 2\,[(a \times b) + (\tfrac{1}{2} \times e \times d)] + 2(c \times a) + (c \times b) + (c \times d) + (c \times e)$$

$$= 2[(4 \times 5) + (\tfrac{1}{2} \times 3 \times 4)] + 2(8 \times 4) + (8 \times 5) + (8 \times 4) + (8 \times 3)$$

$$= 2(20 + 6) + 64 + 40 + 32 + 24$$

$$= 52 + 64 + 40 + 32 + 24$$

$$= 212\text{ cm}^2$$

13.3: Surface Area of a Pyramid

A pyramid is a solid with a polygonal base and slanting sides that meet at a common apex. The surface area of a pyramid is the sum of the area of the slanting faces and the area of the base.

Figure 13.2 shows a right pyramid ABCDE with a square base of side x cm, a slanting height t cm and a perpendicular height h cm. The perpendicular height is usually referred to as the 'height'.

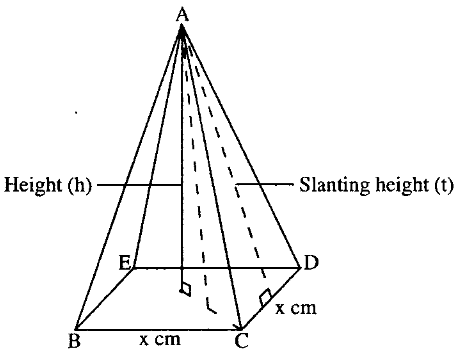


Fig. 13.2

***Example* 2**

Figure 13.3 is a right pyramid with a square base of 4 cm and a slanting edge of 8 cm. Find the surface area of the pyramid.

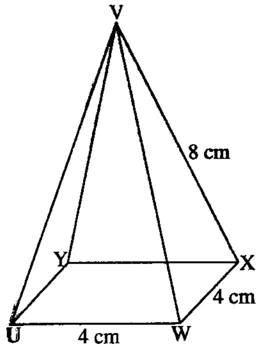


Fig. 13.3

Solution

$$\begin{aligned}
\text{Surface area} &= \text{base area + area of the four rectangular faces} \\
&= (l \times w) + 4(\tfrac{1}{2}bh) \\
&= (4 \times 4) + 4 \times \tfrac{1}{2} \times 4 \times \sqrt{(8^2 - 2^2)} \\
&= 16 + 2 \times 4 \times \sqrt{60} \\
&= 16 + 2 \times 4 \times 7.746 \\
&= (16 + 61.97)\ \text{cm}^2 \\
&= 77.97\ \text{cm}^2
\end{aligned}$$

Example 3

Find the surface area of a pyramid with a rectangular base of 6 cm by 4 cm and a height of 9 cm from the apex to the centre of the base.

Solution

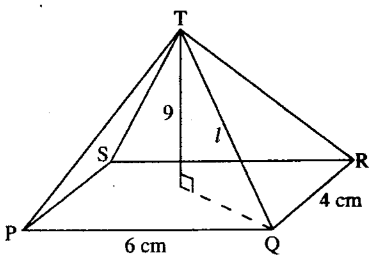


Fig. 13.4

$$\text{Diagonal length} = \sqrt{(6^2 + 4^2)}$$
$$= \sqrt{52}$$
$$= 7.211$$

$$\text{Slanting edge} = \sqrt{(9^2 + 3.606^2)} \qquad (7.211 \div 2 = 3.606 \text{ cm})$$
$$= \sqrt{(81 + 13.00)}$$
$$= \sqrt{94}$$
$$= 9.695 \text{ cm}$$

Height of triangular faces with a base of 4 cm;

$$h = \sqrt{(9.695^2 - 2^2)}$$
$$= \sqrt{(94 - 4)}$$
$$= \sqrt{90}$$
$$= 9.487 \text{ cm}$$

Height of triangular faces with a base of 6 cm;

$$h = \sqrt{(9.695^2 - 3^2)}$$
$$= \sqrt{(94 - 9)}$$
$$= \sqrt{85}$$
$$= 9.220 \text{ cm}$$

$$\text{Surface area} = \underset{\text{base}}{\text{area of}} + \underset{\text{slanting faces}}{\text{area of}}$$
$$= (6 \times 4) + 2(\tfrac{1}{2} \times 4 \times 9.487) + 2(\tfrac{1}{2} \times 6 \times 9.220)$$
$$= 24 + 37.95 + 53.32 \text{ cm}^2$$
$$= 117.3 \text{ cm}^2$$

13.4: Surface Area of a Cone

A cone is a special solid with a circular base.

Project

(i) Draw a circle of radius 9 cm and cut it out.

(ii) Cut out a sector whose arc subtends an angle of 150^0 at the centre, see figure 13.5 (a).

(iii) Find the area of the sector.

(iv) Fold the sector to form the curved surface of an open cone, see figure 13.5 (b).

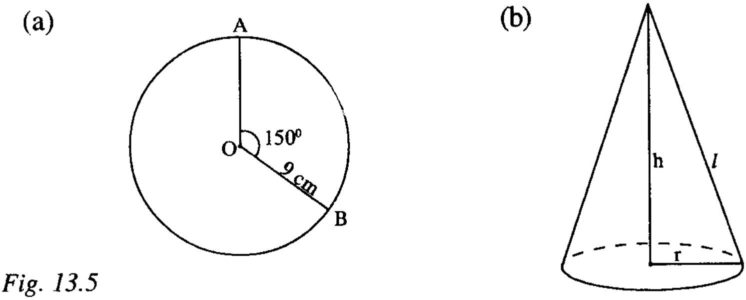


Fig. 13.5

(v) Deduce the slant height l of the cone.
(vi) Deduce the radius (r) of the circular base of the cone.
(vii) Find the area of the curved surface of the cone.
(viii) Compare the area of the curved surface of the cone with the product πrl.
You will notice that:
(i) The area of the curved surface of the cone is equal to the product of πrl.
(ii) The surface area of the closed cone is $\pi r^2 + \pi rl$, where πr^2 is area of the base and πrl the area of the curved surface.

Example 4
Calculate the surface area of a solid cone of base radius 7 cm and height 13 cm.

Solution
Surface area = $\pi r^2 + \pi rl$

$$\text{But } l = \sqrt{13^2 + 7^2} = \sqrt{218} = 14.76 \text{ cm}$$

$$\therefore \text{Surface area} = \left(\tfrac{22}{7} \times 7 \times 7\right) + \left(\tfrac{22}{7} \times 7 \times 14.76\right) = 154 + 324.72 = 478.7 \text{ cm}^2$$

13.5: Surface Area of a Frustum
If a cone or a pyramid is cut through a plane parallel to the base, then the top part forms a smaller cone or pyramid and the bottom part forms a **frustum** (plural frusta). Examples of frusta are a bucket, a lampshade and a hopper.

Project

(i) Make a cone out of a sector of a circle of radius l cm.

(ii) Let R be the radius of the cone so formed and H its height.

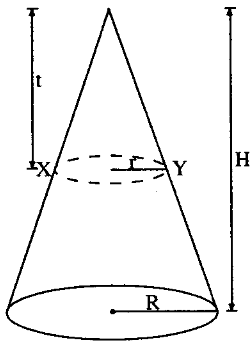


Fig. 13.6

If the cone is cut along plane XY, the lower part of the cone forms a frustum.

From knowledge of similar triangles; $\frac{R}{r} = \frac{H}{t}$ and $\frac{H-t}{R-r} = \frac{t}{r}$

Example 5

Find the surface area of the fabric required to make a lampshade in the shape of a frustum whose top and bottom diameters are 20 cm and 30 cm respectively and height 12 cm. Give your answer to 2 d.p.

Solution

Complete the cone from which the frustum is made, by adding a smaller cone of height x cm.

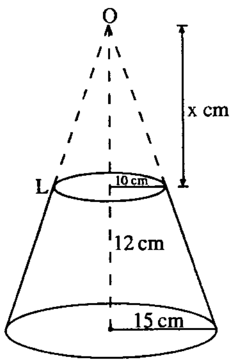


Fig. 13.7

From knowledge of similar triangles;

$$\frac{x}{10} = \frac{x + 12}{15}$$

$15x = 10x + 120$
$15x - 10x = 120$
$5x = 120$
$x = 24$

Surface area of the frustum = area of curved surface of bigger cone − area of curved surface of smaller cone

$$\begin{aligned} \text{Surface area} &= \pi RL - \pi rl \\ &= \tfrac{22}{7} \times 15 \times \sqrt{36^2 + 15^2} - \tfrac{22}{7} \times 10 \times \sqrt{24^2 + 10^2} \\ &= \tfrac{22}{7} \times 15 \times 39 - \tfrac{22}{7} \times 10 \times 26 \\ &= 1838.57\ \text{cm}^2 - 817.14\ \text{cm}^2 \\ &= 1021\ \text{cm}^2 \text{ (4 s.f.)} \end{aligned}$$

13.6: Surface Area of a Sphere

A sphere is a solid that is entirely round with every point on the surface at equal distance from the centre.

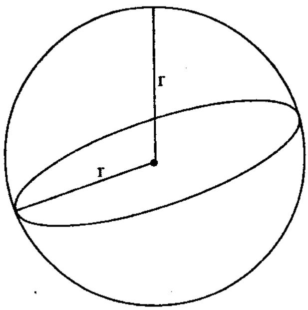


Fig. 13.8

Figure 13.8 is a sphere of radius r cm. The surface area of a sphere of radius r cm is four times the area of a circle of the same radius as the sphere, i.e., surface area is $4\pi r^2$.

Example 6

Find the surface area of a sphere whose diameter is 21 cm.

Solution

Surface area $= 4\pi r^2$

$= 4 \times \frac{22}{7} \times 10.5 \times 10.5$

$= 1\ 386\ \text{cm}^2$

Exercise 13.1

1. Find the surface area of each of the following solids:

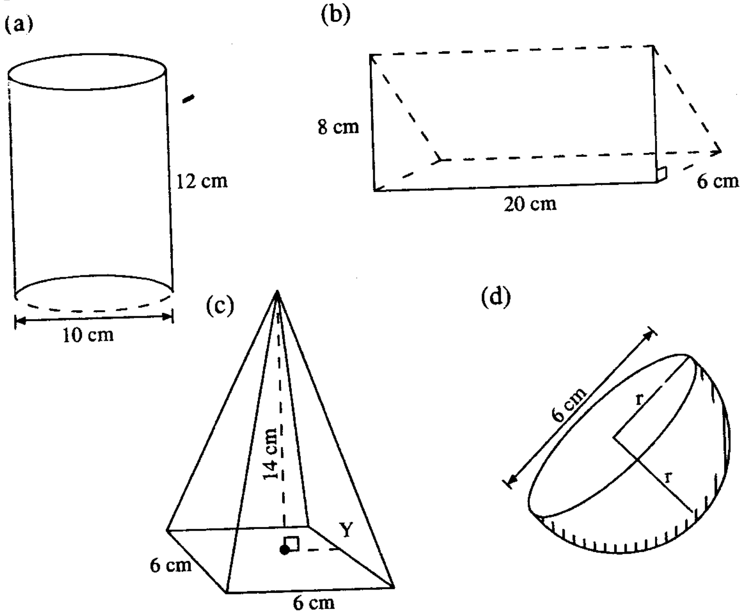


Fig. 13.9

2. Find the surface area of the metal used in making a cylindrical hollow iron pipe 2.1 m long and external diameter 15 cm, given that the metal iron is 1 cm thick. (Give your answer in m^2)
3. A metal storage water tank is in the form of a cylinder with one hemispherical end of radius 2 m. If the other end is flat and the total height of the tank is 8 m and calculate the surface area of the tank.
4. A cone of base diameter 8 cm and height 13 cm is slit and opened out into a sector. Find the angle formed by the two radii of the sector.
5. A right pyramid 6 cm high stands on a rectangular base 6 cm by 4 cm. Calculate the surface area of the pyramid.
6. A spherical ball is 15 cm in diameter. What is its surface area?

7. A right pyramid on a square base 8 metres is 15 metres high.
   (a) Find its surface area.
   (b) If the pyramid is cut at a height of 6 metres from the bottom, find the surface area of the frustum formed.
8. A hopper used in a building construction is a frustum of a right pyramid with a square bottom and a square top of side 2 m and 1 m respectively. If the height of the hopper is 1.5 metres, find its surface area.
9. A water container is a frustum of a cone whose upper and lower radii are 35 cm and 26 cm respectively. If the height of the container is 60 cm, calculate the surface area of the material used to make the container.
10. A scientist using a telescope observed a spherical moving object in space. She established that the object had a radius of 17.5 m. Find the surface area of the object.

Chapter Fourteen

VOLUME OF SOLIDS

Volume is the amount of space occupied by a solid object. The volume of an object is expressed in cubic units.

14.1: Volume of a Prism

A prism is a solid with a uniform cross-section. The volume V of a prism with cross section area A and length l is given by $V = Al$.

Example 1

Find the volume of a triangular-based prism of sides 10 cm, 7 cm and 13 cm and length 25 cm.

Solution

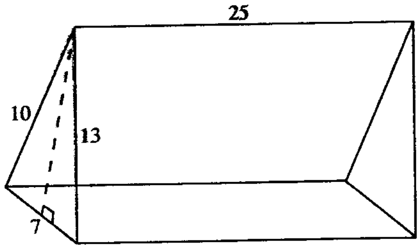


Fig. 14.1

Area of the cross-section, $A = \sqrt{s(s-a)(s-b)(s-c)}$; $s = \frac{30}{2} = 15$

$$= \sqrt{15(15-13)(15-10)(15-7)}$$
$$= \sqrt{15(2)(5)(8)}$$
$$= \sqrt{1\,200}$$
$$= 34.64$$

Volume of the prism $= A \times l$

$$= 34.64 \times 25$$
$$= 866 \text{ cm}^2$$

Example 2

Find the volume of a regular hexagonal nut of side 5 cm and height 3 cm.

Solution

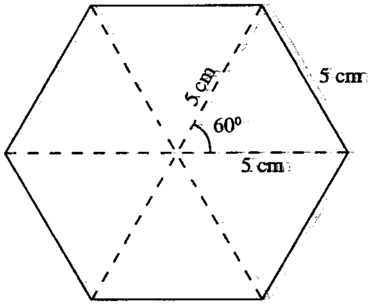


Fig. 14.2

$$\text{Area of cross section} = 6 \times \tfrac{1}{2} \times 5 \times 5 \sin 60 \ (\text{area of triangle} = \tfrac{1}{2}ab\sin\theta)$$
$$= 6 \times \tfrac{1}{2} \times 25 \times \sin 60$$
$$= 64.95$$
$$\text{Volume of the nut} = 64.95 \times 3$$
$$= 194.9 \text{ cm}^3$$

Exercise 14.1

1. A path is 300 m long and 1.8 m wide. Find the volume of gravel needed to cover it to a depth of 5 cm.
2. Calculate the volume of a metal tube whose bore is 75 mm and thickness 9 mm, if it is 5 m long.
3. A water tank of height 2.5 m has a uniform cross-section in the shape of a rectangle with semicircular ends as shown in figure 14.3. The inside length of the tank is 7 m and its width 4 m. Calculate:
   (a) the area of the base.
   (b) the volume of the tank.
   (c) the capacity of the tank in litres.

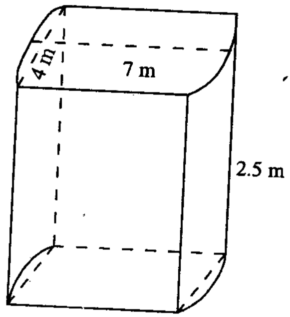


Fig. 14.3

4. A steel metal bar has the shape of a regular hexagon of side 2 cm, and it is 20 m long. Calculate:
   (a) the cross-section area of the metal bar.
   (b) the volume of the metal bar
   (c) the mass in kilograms of the metal bar if 1 m^3 of steel weighs 7 800 kg.

5. A concrete tower on which a water tank is to be placed has a uniform octagonal cross-section of side 3 m. If its height is 24 m, find the volume of concrete used to make it.

6. A metal rod of 20 m length has an isosceles triangular base, where the equal sides are 12 cm each. If the included angle in the base is 40°, calculate:
   (a) the area of the cross-section.
   (b) the volume of the metal rod.
   (c) the mass of the metal rod, if it is made of copper, whose density is 9 000 kg/m^3

7. The cross-section of a container is in the shape of a trapezium. The two parallel sides of the container are 3.6 m and 2.1 m long, while the perpendicular height between them is 2.4 m. If its length is 6 m, find the volume of the container.

8. Figure 14.4 represents a swimming pool. Calculate:
   (a) the volume of the pool in cubic metres.
   (b) the capacity of the pool in litres.

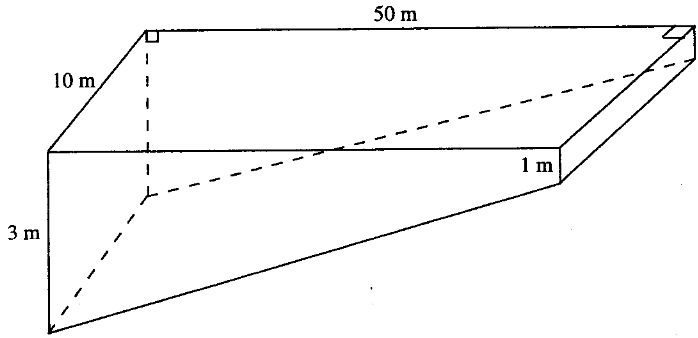


Fig. 14.4

14.2: Volume of a Pyramid

Project

Figure 14.5 shows the net of a square based pyramid. (All measurements are in centimetres)

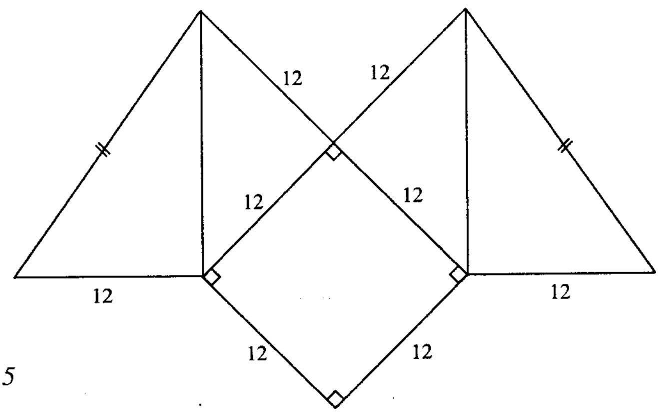


Fig. 14.5

Construct three such nets and fold them to form three pyramids. Assemble them to make a cube.

(i) What is the volume of the cube you have made from the pyramids?

(ii) What is the volume of one pyramid?

You should notice that:

(i) The volume of the cube is $(12 \times 12 \times 12)\ cm^3 = 1\ 728\ cm^3$.

(ii) the volume of the each pyramid is $\frac{1}{3}$ of the volume of the cube. That is;

volume of pyramid $= \frac{1}{3}(12 \times 12 \times 12)\ \text{cm}^3$

$= 576\ \text{cm}^3$

In other words, the volume of each pyramid is given by;

$\frac{1}{3}$ **(base area of pyramid × height)**

In general, the volume (V) of any pyramid is given by;

$V = \frac{1}{3}$ **(base area) × height**

$= \frac{1}{3}Ah,$

where A is the area of the base and h the perpendicular height of the pyramid.

Example 3

Find the volume of the pyramid in figure 14.6:

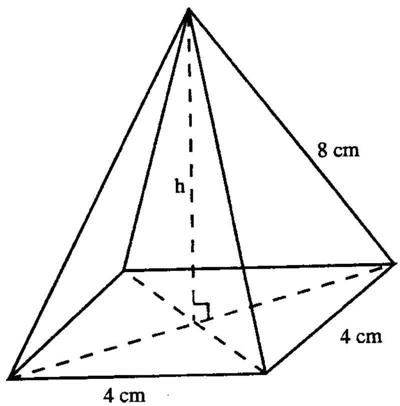


Fig. 14.6

Solution

Area of base $= 4 \times 4$

$= 16\ \text{cm}^2$

To find the height of the pyramid, proceed as follows:

Half the length of the diagonal $= \frac{1}{2} \times \sqrt{(4^2 + 4^2)}$

$= \frac{1}{2}\sqrt{32}$

$= \frac{1}{2} \times 5.657$

$= 2.829\ \text{cm}$

$\therefore$ Volume $= \frac{1}{3} \times \frac{22}{7} \times 25 \times 12\ \text{cm}$

The height, h, of the pyramid $= \sqrt{(8^2 - 2.829^2)}$
$= \sqrt{64 - 8.003}$
$= 7.483$ cm

Therefore, the volume of the pyramid $= \frac{1}{3} \times 16 \times 7.483$
$= \frac{1}{3} \times 119.7$
$= 39.91\ cm^3$

14.3: Volume of a Cone

Since a cone is a right pyramid with a circular base, its volume V is given by;

$V = \frac{1}{3}$ (base area) $\times$ height

$= \frac{1}{3} \pi r^2 h$

Example 4

Calculate the volume of a cone whose height is 12 cm and length of the slant height 13 cm. (Take $\pi = \frac{22}{7}$)

Solution

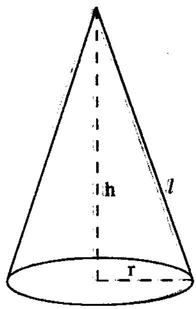


Fig. 14.7

Volume $= \frac{1}{3}$(base area) $\times$ height

$= \frac{1}{3} \pi r^2 h$

But, base radius r $= \sqrt{13^2 - 12^2}$
$= \sqrt{25}$
$= 5$

$$\therefore \text{Volume} = \frac{1}{3} \times \frac{22}{7} \times 25 \times 12 \text{ cm}$$
$$= 314.3 \text{ cm}^3$$

Example 5

Figure 14.7 represents a frustum of base radius 2 cm and height 3.6 cm. If the height of the cone from which it was cut was 6 cm, calculate:

(a) the radius of the top surface.

(b) the volume of the frustum.

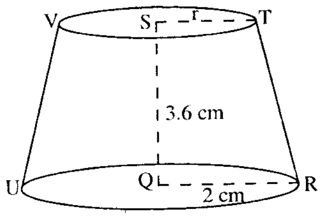


Fig. 14.8

Solution

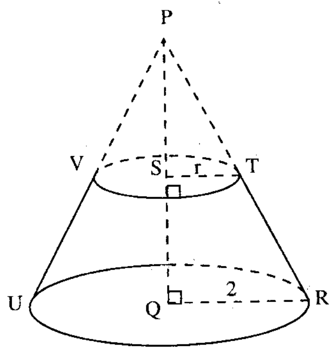


Fig. 14.9

(a) Triangles PST and PQR are similar.

$$\therefore \frac{PQ}{PS} = \frac{QR}{ST} = \frac{PR}{PT}$$

So, $\frac{6}{2.4} = \frac{2}{ST}$

ST $= 0.8$ cm

$\therefore$ The radius of the top surface is 0.8 cm

(b) Volume of the frustum = volume of large cone – volume of smaller cone

$= \frac{1}{3} \times 3.142 \times 4 \times 6 - \frac{1}{3} \times 3.142 \times (0.8)^2 \times 2.4$
$= 25.14 - 1.61$
$= 23.53 \text{ cm}^3$

Alternatively

Using volume scale factor;

Linear scale factor (LSF) $= \frac{2}{0.8}$
$= 2.5$

Volume scale factor (VSF) $= \left(\frac{5}{2}\right)^3$
$= \frac{125}{8}$

Volume of the large cone $= 25.14 \text{ cm}^3$

Volume of the small cone $= 25.14 \times \frac{8}{125}$
$= 1.61 \text{ cm}^3$

Volume of the frustum = volume of large cone – volume of small cone
$= 25.14 - 1.61$
$= 23.53 \text{ cm}^3$

Exercise 14.2

1. Find the volume of each of the following pyramids:
   (a) Triangular base, sides 5 cm, 12 cm, and 13 cm, height 24 cm.
   (b) Square base, side 6 cm, height 10 cm.
   (c) Rectangular base, 8 cm by 10 cm, height 12 cm.
2. Find the volume of each of the following right pyramids:
   (a) Rectangular base, 18 cm by 24 cm, slant edge 39 cm.
   (b) Square base, side 8 cm, edge 8 cm.
   (c) Triangular base, sides 10 cm, 12 cm and 4 cm, height 24 cm.
3. (a) The volume of a pyramid is 120 cm^3. Calculate its height and the surface area if the base is a square of side 6 cm.
   (b) Find the height of a pyramid with base area 24 cm^2 and volume 96 cm^3.
4. Calculate the height and the length of the slant edge of a right pyramid whose base is a square of side 5 cm and volume 125 cm^3.
5. The volume of a right pyramid of height 16 cm on a square base is 192 cm^3. Calculate the length of the side of the base.
6. VABCD is a right pyramid with a square base ABCD. Calculate its volume if the height is 12 cm and the slant edge is 16 cm.
7. Figure 14.10 shows the net of a pyramid with a square base of side 32 cm. The other four faces are isosceles triangles whose equal sides are 34 cm each. Calculate the volume of the pyramid.

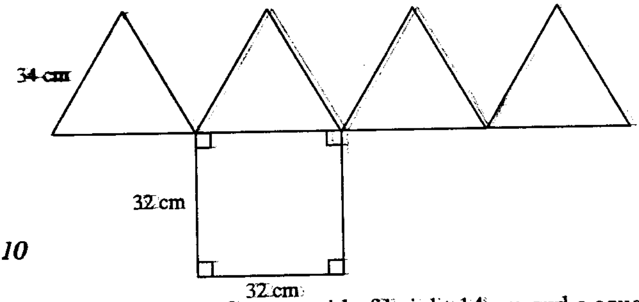


Fig. 14.10

8. Calculate the volume of a pyramid of height 14 cm and a square base of side 24 cm.
9. A cone has a base radius of 9.44 cm and a volume of 113.6 cm^3. Calculate the height of the cone.
10. Find the volume of each of the following cones:
    (a) Base area 18 cm^2, height 6 cm.
    (b) Slant height 15 cm, height 9 cm.
    (c) Base perimeter 16 cm, height 10 cm.
    (d) Slant height 17 cm, height 7 cm.
    (e) Base radius 9 cm, height 24 cm.
11. Calculate the total surface area and volume of a frustum of a pyramid whose ends are squares of 10 cm and 12 cm and the height between the two ends is 4 cm.
12. A toy consists of a conical top and a cylindrical base. The diameter of the base is 5 cm and the height of the cylindrical part is 4 cm. If the total height of the toy is 9 cm, calculate:
    (a) its total surface area.
    (b) its total volume.
13. A frustum is cut from a cone whose radius is r cm and height h cm. If the height of the frustum is t centimetres, express the volume of the frustrum as a fraction of the volume of the cone.
14. Figure 14.11 shows a triangle in which AB = 18 cm, BC = 6 cm, BD = 7 cm and DE is parallel to BC. If the triangle is rotated about AB, calculate:
    (a) the surface area of the cone formed.
    (b) the volume of the cone.
    (c) the volume of the frustum formed by the portion BCED.

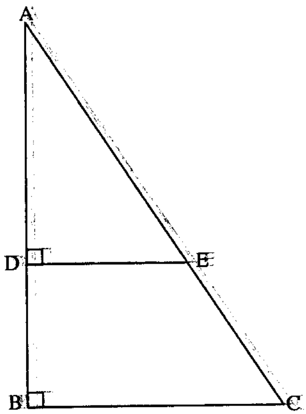


Fig. 14.11

14.4: Volume of a Sphere

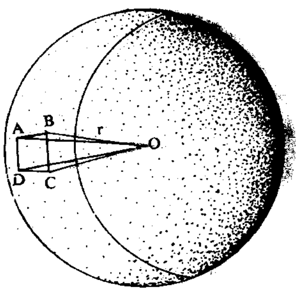


Fig. 14.12

Figure 14.12 shows a sphere with radius r cm and a unit square ABCD. Let the volume of pyramid OABCD be V cm^3.

Therefore, $V = \frac{1}{3}(\text{base area} \times \text{height})$

$= \frac{1}{3} \times 1 \times r$

$= \frac{r}{3}\ cm^3$

Surface area of the sphere $= 4\pi r^2\ cm^2$

Therefore; volume of the sphere = volume of each pyramid × surface area of a sphere

$= \frac{r}{3} \times 4\pi r^2$

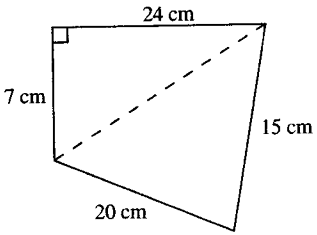
$$= \frac{4\pi r^3}{3} \text{ cm}^3$$

Exercise 14.3

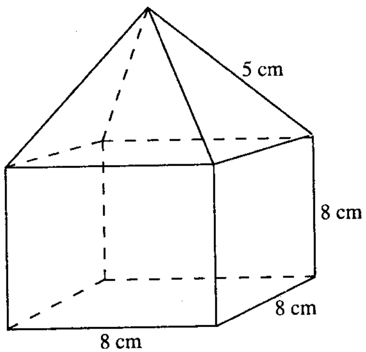
1. Calculate the volume of a sphere whose radius is:
   (a) 4.8 cm. (b) 21 cm.
2. A solid metal cylinder is 60 cm long and 18 cm in diameter. How many solid spheres of diameter 8 cm could be moulded from such a cylinder?
3. The internal and external radii of a spherical shell are 8 cm and 9 cm respectively. Calculate the volume of the material of the shell to the nearest cm^3.
4. A cylindrical tank of radius 0.6 m contains water to a depth of 1 m. A solid metal sphere of radius 0.5 m is placed in it. Calculate the rise of the water level in centimetres.
5. The volume of water in a measuring cylinder is 25.2 cm^3. After a solid metal sphere is immersed into it, the measuring cylinder reads 29.4 cm^3. Calculate the radius of the sphere.
6. The mass of 1 cm^3 of lead is 11.4 g. Calculate the mass of a lead sphere whose radius is 15 cm.
7. If 1 cm^3 of copper metal has a mass of 8.88 g, calculate the mass of a quarter sphere of copper whose radius is 24 cm.
8. A spherical container which is 30 cm in diameter is $\frac{3}{4}$ full of water. The water is emptied into a cylindrical container of diameter 12 cm. What is the depth of the water in the cylindrical container?
9. A solid copper sphere of 36 cm is to be moulded from a copper wire 0.8 mm in diameter. How many metres of such wire is required to make the sphere?

Mixed Exercise 2

1. Find the area of the quadrilateral below:

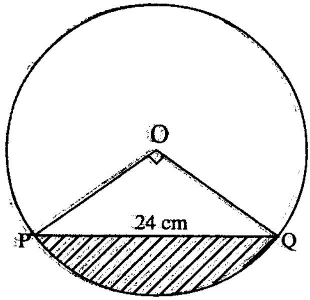


2. The length of an arc of a circle is $\frac{1}{10}$ of its circumference. If the area of the circle is $13.86\ \text{cm}^2$, find:
   (a) the angle subtended by the arc at the centre of the circle.
   (b) the area of the sector enclosed by this arc.
3. A model of a tent consists of a cube and a pyramid on a square base, as below:

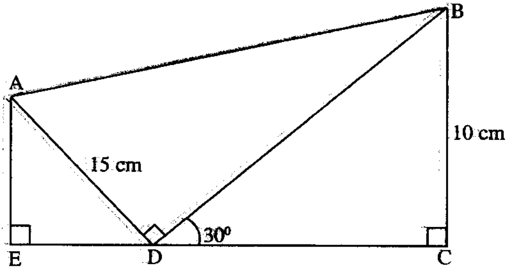


   Calculate the total surface area of the model.
4. Solve the equation a**m** + b**h** = **r**, given that $\mathbf{m} = \begin{pmatrix}-3\\-2\end{pmatrix}$, $\mathbf{n} = \begin{matrix}0\\4\end{matrix}$.

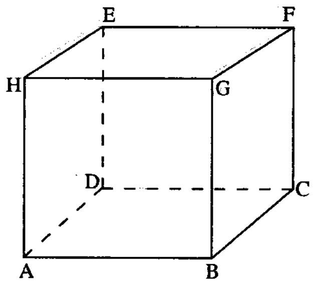
   $\mathbf{r} = \begin{pmatrix}-6\\0\end{pmatrix}$ and a and b are scalars.
5. Find the volume of a prism 15 cm long and whose cross-section is a regular heptagon of side 10 cm.
6. In the figure below, 0 is the centre of the circle, $\angle POQ = 90^0$ and PQ = 24 cm: Find:
   (a) the area of the sector POQ.
   (b) the area of the shaded region.



7. Find the area of the figure below if AD = 15 cm, BC = 10 cm, $\angle AED = \angle ADB = \angle BCD = 90°$ and $\angle BDC = 30°$:



8. John walks 4.8 km due north from point A to B and then covers a further 6 km due east to C. What is the shortest distance from A to C?
9. The figure below is a cube of side 6 cm:



Calculate:

(a) the distance AC (b) the distance AG (c) the distance AF

10. The area of a rhombus is 442 cm^2. If one of its diagonals is 34 cm long, what is:

(a) the length of the other diagonal?
(b) the length of the side of the rhombus?

11. The base of a right pyramid is a regular hexagon of side 8 cm. If the slant edges are 10 cm, what is the:
(a) surface area of the pyramid? (b) volume of the pyramid?

12. The cross-section of a prism is a triangle with sides 3 cm, 5 cm and 7 cm. If the prism is 12 cm long, what is its volume?

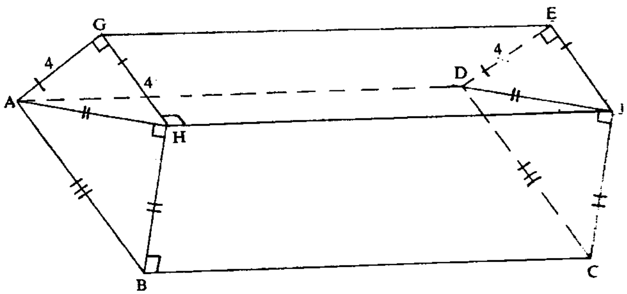
13. Given that cos x = sin (3x + 10), find:
(a) x. (b) tan x.

14. A television aerial is held upright by a wire 16 m long which is fixed to the top of the aerial and to the ground 3 metres from the foot of the aerial. What is the inclination of the wire to the ground?

15. A frustum of height 6 cm is cut from a pyramid whose base is a square of side 20 cm and height 14 cm. Find the volume of the frustum.

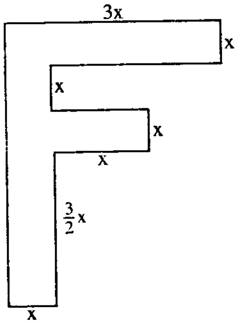
16. The radius and height of a cylinder are 3.5 cm and 6.5 cm respectively. Calculate the surface area, the height and the radius of a similar cylinder whose volume is 2 002 cm^3. (Take $\pi = \frac{22}{7}$)

17. In the figure below, AG = GH = EJ = ED = 4 cm, AH = BH = CJ = DJ and AB = CD.



Calculate:
(a) CJ and DC.
(b) The ratio of the area of AGH to ABH.
(c) The ratio of the volume of AHJDEG to ABCDJH.
(d) the volume and the surface area of the whole block if GE = BC = 2DJ.

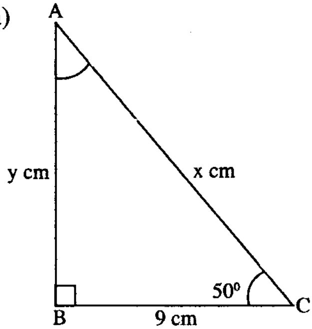
18. The area of the figure below is A cm^2 and its perimeter is P cm. If A = P + 8, find x.



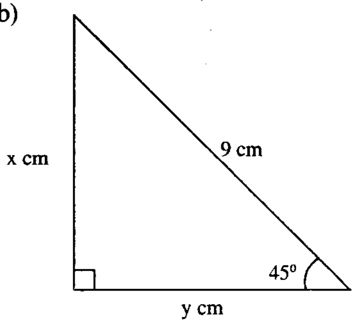
19. Find the surface area of a cone whose height is 5.2 cm and volume 49 cm^3.
20. A boy's shadow is 2.2 m long when the angle of elevation is 30°. Calculate the height of the boy.
21. Calculate the volume of a cone whose base radius is 5 cm and curved surface area is 109.9 cm^2.
22. Find the length of a side of a regular pentagon whose vertices lie on the circumference of a circle of radius 5 cm.
23. A man walks 1 km on a bearing of 140^0. How far is he:
    (a) east and of the starting point?
    (b) south of the starting point?
24. A triangle ABC is such that $\angle BCA = 90^0$, BC = 48 cm and $\angle ABC = 60^0$. Determine lengths AB and AC.
25. An observer stationed 20 m away from a tall building finds that the angle of elevation of the top of the building is 68^0 and the angle of depression of its foot is 50^0. Calculate the height of the building.
26. Calculate the volume of metal required to make a hemispherical bowl with internal and external radii 8.4 cm and 9.1 cm respectively.
27. Onyango observed a hawk exactly overhead at the same time as Kamau observed it at an angle of elevation of 50^0. If Onyango and Kamau were 25 m apart on a level ground, how high was the hawk above the ground, given that Kamau's eye level was 172 cm above the ground?
28. A straight road rises steadily at an angle of 4^0 to the horizontal. If a car travels 2 km up the road, calculate its vertical and horizontal displacement.
29. A pyramid PQRST has a rectangular base QRST and height h. If the base of the pyramid measures 8 cm by 4 cm and perpendicular height is equal to the diagonal of the base, find its volume and surface area.
30. A ladder 8 m long is leaning against a vertical wall. The ladder makes an angle of 70^0 with the level ground. Calculate the distance of the top of the ladder from the ground.

31. Find the value of the unknown lengths in each of the following diagrams:

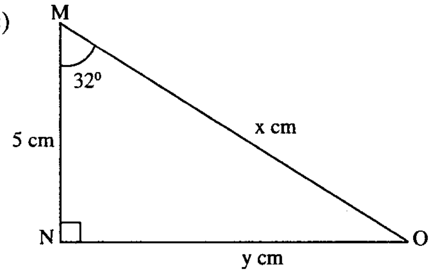
(a)



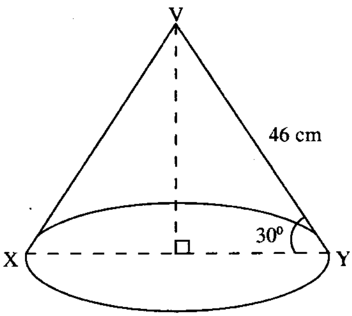
(b)



(c)



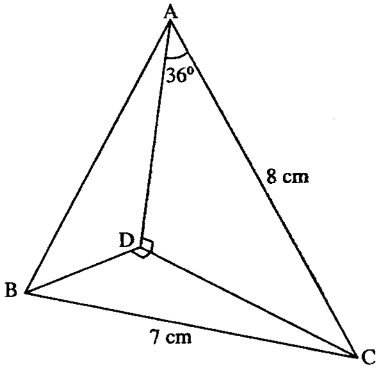
32. The figure below shows a cone in which angle VYX = 30^0 and VY = 46 cm. Calculate the height and diameter of the cone.



33. AB is a chord of a circle with centre O and radius 12.3 cm. If $\angle OAB = 70^0$, calculate the length AB and the distance of O from AB.
34. Find the radius of a cone whose height is 8 cm and volume 115 cm^3.
35. The two parallel sides of a trapezium measure 12 cm and 9 cm. A third side measures 10.4 cm and makes an angle of 50^0 with the longest side. Calculate the area of the trapezium.

36. A bucket is 44 cm in diameter at the top and 24 cm in diameter at the bottom. Find its capacity in litres if it is 36 cm deep.

37. The figure below shows a tetrahedron ABCD in which AC = 8 cm and BC = 7 cm. AD and BD are perpendicular to CD and $\angle CAD = 36^0$.



(a) Calculate: (i) AD. (ii) CD. (iii) BD. (iv) $\angle BCD$.

(b) Find AB, given that $\angle ADB = 120^0$ and $\angle DAB = 20^0$.

38. Find the radius of a hemispherical bowl which can hold 0.5 litres of liquid.

39. The dimensions, in centimetres, of a rectangle are $(2n + 3)$ by $(n + 1)$ and its area is 817 cm^2. Determine the length of the diagonal.

40. A prism has a regular heptagonal cross-section of radius 91 cm. If the prism has a length of 120 cm, find its surface area.

Chapter Fifteen

QUADRATIC EXPRESSIONS AND EQUATIONS

15.1: Expansion

We have already seen how to remove brackets in expressions of the form; $a(x + y)$, that is, $a(x + y) = ax + ay$.

The same idea can be used to remove brackets in expressions such as $(x + 5)(3x + 2)$.

We let $3x + 2$ be a;

Therefore, $(x + 5)(3x + 2) = (x + 5)a$
$= xa + 5a$

Thus, $xa + 5a = x(3x + 2) + 5(3x + 2)$
$= 3x^2 + 2x + 15x + 10$
$= 3x^2 + 17x + 10$

When the expression $(x + 5)(3x + 2)$ is written in the form $3x^2 + 17x + 10$, it is said to have been **expanded**.

Thus, $3x^2 + 17x + 10$ is the expanded form of the expression $(x + 5)(3x + 2)$.

Example 1

Expand $(m + 2n)(m - n)$.

Solution

Let $(m - n)$ be a.

$$
\begin{aligned}
\text{Then, } (m + 2n)(m - n) &= (m + 2n)a \\
&= ma + 2na \\
&= m(m - n) + 2n(m - n) \\
&= m^2 - mn + 2mn - 2n^2 \\
&= m^2 + mn - 2n^2
\end{aligned}
$$

Example 2

Expand $(\frac{1}{4} - \frac{1}{x})^2$.

Solution

$$
\begin{aligned}
(\tfrac{1}{4} - \tfrac{1}{x})^2 &= (\tfrac{1}{4} - \tfrac{1}{x})(\tfrac{1}{4} - \tfrac{1}{x}) \\
&= \tfrac{1}{4}(\tfrac{1}{4} - \tfrac{1}{x}) - \tfrac{1}{x}(\tfrac{1}{4} - \tfrac{1}{x}) \\
&= \frac{1}{16} - \frac{1}{4x} - \frac{1}{4x} + \frac{1}{x^2} \\
&= \frac{1}{16} - \frac{1}{2x} + \frac{1}{x^2}
\end{aligned}
$$

Exercise 15.1

Expand and simplify each of the following expressions:

1. (a) $(6x + 2)(4x + 3)$ (b) $(3x - 2)(x + 5)$
   (c) $(x - 1)(x + 2)$ (d) $(2x - 3)(x + 2)$
   (e) $(6x - 2)(2x + 8)$ (f) $(2 - x)(x + 4)$
2. (a) $(x + 7)(x + 8)$ (b) $(3x - 2)(2x + 3)$
   (c) $(5x + 3)(3x + 5)$ (d) $(7x - 4)(4x - 7)$
   (e) $(m - n)(n - m)$ (f) $(2x - 3)(x + 3)$
3. (a) $(ax + b)(2ax - 3b)$ (b) $(4 - 2x)\left(\frac{1}{2} - x\right)$
   (c) $(6x - 5)(5x + 6)$ (d) $(y - 2)(y + 2)$
   (e) $(x + 2a)(x + 2a)$ (f) $(3x - 4)(5x - 6)$
4. (a) $(x + 2a)(x - 2a)$ (b) $(x - 7)(x + 7)$
   (c) $(x + 2)^2$ (d) $(4x + 2)^2$
5. (a) $\left(\frac{1}{2} + x\right)^2$ (b) $(ax + d)^2$
   (c) $\left(\frac{1}{4} + \frac{1}{x}\right)^2$ (d) $(x - 3)^2$
   (e) $\left(4x - \frac{3}{4}\right)^2$ (f) $\left(1 - \frac{1}{x}\right)^2$

15.2: The Quadratic Identities

From the previous exercise, you may have noticed that:

$(x + 2)^2 = x^2 + 4x + 4$

$(x - 3)^2 = x^2 - 6x + 9$, and

$(x + 2a)(x - 2a) = x^2 - 4a^2$

In general;

(i) $(a + b)^2 = a^2 + 2ab + b^2$

(ii) $(a - b)^2 = a^2 - 2ab + b^2$

(iii) $(a + b)(a - b) = a^2 - b^2$ (difference of two squares)

The three identities above are useful in expansion and factorisation of many quadratic expressions.

Exercise 15.2

Use the above identities to write down the expansions of each of the following expressions:

1. (a) $(x + 5)^2$ (b) $(x - 5)^2$
   (c) $(4x + 3)^2$ (d) $(2x + 3)(2x + 3)$

2. (a) $(3-x)^2$ (b) $\left(\frac{1}{2}+x\right)\left(\frac{1}{2}+x\right)$

(c) $\left(\frac{1}{x}-1\right)^2$ (d) $\left(\frac{1}{4}+\frac{3}{4}b\right)^2$

3. (a) $\left(\frac{1}{2}a-\frac{1}{3}b\right)^2$ (b) $\left(1-\frac{1}{a}\right)\left(1+\frac{1}{a}\right)$

(c) $\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right)$ (d) $(-2x-3y)(-2x+3y)$

4. (a) $\left(\frac{1}{3y}+\frac{1}{4x}\right)\left(\frac{1}{3y}+\frac{1}{4x}\right)$ (b) $\left(-\frac{1}{3}x-1\right)^2$

(c) $\left(\frac{1}{5}a+\frac{4}{5}b\right)^2$ (d) $\left(\frac{3}{7}x-\frac{1}{2}y\right)^2$

15.3: Factorisation

When the Coefficient of x^2 is One

The expression $ax^2 + bx + c$, where a, b and c are constants, is known as a **quadratic expression**. In this expression, a is called the coefficient of x^2, b the coefficient of x and c the constant term. Consider the following expressions:

(i) $(x+2)(x+5) = x^2 + 2x + 5x + 2 \times 5$
$= x^2 + 7x + 10$

(ii) $(x+4)(x+3) = x^2 + 3x + 4x + 4 \times 3$
$= x^2 + 7x + 12$

(iii) $(x+5)(x-2) = x^2 + 5x - 2x + 5x(-2)$
$= x^2 + 3x - 10$

(d) $(x-5)(x-2) = x^2 - 5x - 2x + -5 \times (-2)$
$= x^2 - 7x + 10$

We note from the examples that:

(i) $x^2 + 7x + 10$ can be factorised into $(x+2)(x+5)$

(ii) $x^2 + 7x + 12$ can be factorised into $(x+4)(x+3)$, etc.

In each case:

(i) the sum of the constant terms of the factors is equal to the coefficient of x in the expression.

(ii) the product of the constant terms of the factors is equal to the constant term of the expression.

Example 3

Factorise the expressions:

(a) $x^2 + 6x + 5$

(b) $x^2 + x - 12$

Solution

In both cases, the factors will be of the form $(x + a)(x + b)$.

(a) $x^2 + 6x + 5$

Look for whole numbers a and b such that;

$a + b = 6$ (sum), and

$a \times b = 5$ (product)

In this case, 1 and 5 are the numbers.

$$
\begin{aligned}
x^2 + 6x + 5 &= x^2 + x + 5x + 5 \\
&= x(x + 1) + 5(x + 1) \\
&= (x + 1)(x + 5)
\end{aligned}
$$

(b) $x^2 + x - 12$

Think of two integers whose sum is 1 and product is -12.

These are 4 and -3.

$$
\begin{aligned}
\text{Then, } x^2 + x - 12 &= x^2 - 3x + 4x - 12 \\
&= x(x - 3) + 4(x - 3) \\
&= (x - 3)(x + 4)
\end{aligned}
$$

Find the numbers whose sum is the first number of the pair and whose product is second in each of the following:

(i) (5, 6) (ii) (13, 42) (iii) (1, –20)

(iv) (5, –6) (v) (–5, 6) (vi) (–7, 6)

(vii) (3, –40) (viii) (8, –20) (ix) (1, –12)

(x) (–12, 35)

Exercise 15.3

Factorise each of the following expressions:

1. (a) $x^2 + 6x + 8$ (b) $x^2 - 5x + 6$ (c) $x^2 + 4x - 21$
 (d) $x^2 + x - 2$ (e) $x^2 + 2x - 35$ (f) $x^2 - 10x + 24$
2. (a) $x^2 - 4x - 32$ (b) $x^2 + 3x - 54$ (c) $x^2 + 2x + 1$
 (d) $x^2 + 4x + 4$ (e) $x^2 - 2x + 1$ (f) $x^2 - 14x + 49$
3. (a) $x^2 - 16x + 64$ (b) $x^2 - 4$ (c) $1 - x^2$
 (d) $2x^2 - 32$ (e) $3x^2 - 15x + 18$ (f) $x^2 - 5x - 6$
4. (a) $x^2 + 2x + ax + 2a$ (b) $t^3 + 8t^2 + 12t$

When the Coefficient of x^2 is Greater than One

In this section, we shall deal with the general quadratic expression of the form $ax^2 + bx + c$, where a is greater than 1.

Consider the following expansions:

(i) $(4x + 3)(2x + 1) = 8x^2 + 4x + 6x + 3$
$= 8x^2 + 10x + 3$

(ii) $(3x + 2)(x - 3) = 3x^2 - 9x + 2x + 2x - 6$
$= 3x^2 - 7x - 6$

(iii) $(3x + 2)(2x - 5) = 6x^2 - 15x + 4x + 2 \times -5$
$= 6x^2 - 11x - 10$

(iv) $(3x - 2)(2x - 3) = 6x^2 - 9x - 4x + (-2x - 3)$
$= 6x^2 - 13x + 6$

In the examples above, we multiplied out factors to get the final expressions on the right hand side. If instead we are given the expressions on the right hand side, how do we get their factors?

Consider the expression $8x^2 + 10x + 3$ [(i) above]. It can be factorised as follows;

— Look for two numbers such that:
- their product is $8 \times 3 = 24$.
  Note that 8 is the coefficient of x^2 and 3 is the constant term in the given expression.
- Their sum is 10, where 10 is the coefficient of x.

The numbers are 4 and 6.

— Rewrite the term 10x as 4x + 6x.
Thus, $8x^2 + 10x + 3 = 8x^2 + 4x + 6x + 3$

— Use the grouping method to factorise the expression.
Thus, $8x^2 + 4x + 6x + 3 = 4x(2x + 1) + 3(2x + 1)$
$= (4x + 3)(2x + 1)$

In the examples (ii), (iii) and (iv), give for each the product and the sum of the two required numbers.

Example 4

Factorise $6x^2 - 13 + 6$.

Solution

To factorise the expression $6x^2 - 13x + 6$, we look for two numbers such that:

(i) the product is $6 \times 6 = 36$.

(ii) their sum is -13.

The numbers are -4 and -9.

Therefore, $6x^2 - 13x + 6 = 6x^2 - 4x - 9x + 6$
$= 2x(3x - 2) - 3(3x - 2)$
$= (2x - 3)(3x - 2)$

Generally, **to factorise the expression $ax^2 + bx + c$, we look for two numbers such that their product is ac and their sum is b.**
We split the middle term into two terms such that their coefficients are the numbers so determined.

Exercise 15.4

Factorise each of the following expressions:

1. (a) $2x^2 + 3x - 2$ (b) $3x^2 - 2x - 8$
   (c) $4x^2 + 7x + 3$ (d) $5x^2 - 21x + 4$
2. (a) $14x^2 - 16x + 2$ (b) $3x^2 + 11x + 6$
   (c) $8 - 2x - 3x^2$ (d) $16x + 24 + 2x^2$
3. (a) $35x^2 + 43x + 12$ (b) $-21x^2 + 58x - 21$
   (c) $150x^2 - 25x - 21$ (d) $17x^2 + 82x - 15$
4. (a) $8x^2 + 6x + 1$ (b) $x^2 - 4x - 117$
   (c) $9x^2 + 48x + 64$ (d) $75x^2 + 10x - 21$
5. (a) $1 + 7x - 30x^2$ (b) $19x^2 - 22x + 3$
   (c) $63x^2 + 130x + 63$ (d) $x^2 - \frac{2}{15}x - \frac{1}{15}$

In questions 6 to 10, use any suitable method to factorise the given expressions where possible.

6. (a) $4x^2 - 12x + 9$ (b) $9x^2 + 6x + 1$
   (c) $2x^2 - 32y^2$ (d) $1 - \frac{2}{y} + \frac{1}{y^2}$
7. (a) $t^8 - t^4$ (b) $35x^2 - x - 12$
   (c) $4x^2 + 2x + \frac{1}{4}$ (d) $2x^2 + 4x + 1$
8. (a) $21x^2 + 17x - 4$ (b) $25x^2 - \frac{5}{2}x + \frac{1}{16}$
   (c) $26x - 29x^2 + 3$ (d) $\frac{1}{16}x^2 - \frac{7}{2}x + 49$
9. (a) $24x^2 - 43x - 56$ (b) $33x^2 - 16x - 4$
   (c) $\frac{1}{x^2} - \frac{4}{x} + 4$ (d) $w^4 - t^4 + t^2 + w^2$
10. (a) $25x^2 + 10xy + y^2$ (b) $(x + y)^2 + (x + y) + 1$
    (c) $4x^2 + 12x + 9$ (d) $6x^2 + 6x + 1$

16.4 : Quadratic Equations

Any equation of the form $ax^2 + bx + c = 0$ is known as a **quadratic equation.** In this section, we shall look at solutions of quadratic equations using the factor method.

Example 5

Solve $x^2 + 3x - 54 = 0$.

Solution

Factorising the L.H.S.

$x^2 + 3x - 54 = x^2 - 6x + 9x - 54 = 0$

$x(x - 6) + 9(x - 6) = 0$

$(x - 6)(x + 9) = 0$

In general, if $a \times b = 0$, either $a = 0$ or $b = 0$.

Then, either $x + 9 = 0$ or $x - 6 = 0$

Therefore, $x = -9$ or $x = 6$

Check the answers by substituting into the given equation.

Example 6

Solve the equation $4x^2 + 7x + 3 = 0$.

Solution

Factorising L.H.S., $4x^2 + 7x + 3 = (4x^2 + 4x) + (3x + 3) = 0$

$4x(x + 1) + 3(x + 1) = 0$

$(x + 1)(4x + 3) = 0$

Either $(x + 1) = 0$ or $(4x + 3) = 0$

Therefore, $x = -1$ or $x = -\frac{3}{4}$

Numbers that satisfy an equation (its solutions) are called the **roots** of the equation. In example 5, the roots of $x^2 + 3x - 54 = 0$ are -9 and 6.

Exercise 15.5

Solve each of the following equations:

1. (a) $x^2 + 6x + 8 = 0$ (b) $2x^2 + 3x - 2 = 0$
   (c) $14x^2 - 16x + 2 = 0$ (d) $4x^2 + 7x - 5 = x^2 - 9$
2. (a) $x^2 - 25 = 0$ (b) $3x^2 - 2x = 5$
   (c) $3x^3 + x^2 - 4x = 0$ (d) $1 - 6x^2 = x$
3. (a) $4x^2 - x - \frac{1}{2} = 0$ (b) $x + \frac{20}{x} = 9$
   (c) $16x + 24 + 2x^2 = 0$ (d) $\frac{1}{4} - \frac{1}{9}x^2 = 0$
4. (a) $\frac{3}{x^2} - \frac{8}{x} = 16$ (b) $(2x - 1)^2 - 1 = 3$
   (c) $x^2 + \frac{5}{6}x = \frac{1}{6}$ (d) $\left(1 - \frac{1}{x}\right)^2 - 1 = 35$
5. (a) $\frac{1}{3}x^2 = 2x - 3$ (b) $x^2 - \frac{1}{25} = \frac{2}{5}$
   (c) $-\frac{12}{x} = -x + 4$ (d) $x^2y^2 - x^2 - y^4 + y^2 = 0$

6. (a) $7x^2 - 15x + 8 = 0$ (b) $8x^2 + 6x + 1 = 0$

(c) $x^2 - \frac{2}{15}x - \frac{1}{15} = 0$ (d) $\frac{2x^2 - 15}{x} = 7$

7. (a) $\frac{x^2}{2} + 1 = \frac{4.5x}{2}$ (b) $1 - x = \frac{1}{x} + 3$

(c) $x^2 - \frac{1}{18} = \frac{x}{6}$ (d) $4\left(\frac{1}{y^2}\right) - 4\left(\frac{1}{y}\right) - 1 = 0$

8. (a) $\frac{6}{x - 4} = \frac{x + 4}{x}$ (b) $\frac{6x + 8}{2} + \frac{1}{x} = 0$

15.5: Formation of Quadratic Equations

From Given Roots

Given that the roots of a quadratic equation are $x = 2$ and $x = -3$, how do we obtain the equation?

To obtain the equation, we proceed as follows:

If $x = 2$, then $x - 2 = 0$.

If $x = -3$, then $x + 3 = 0$.

Therefore, $(x - 2)(x + 3) = 0$.

The required equation is $x^2 + x - 6 = 0$.

Exercise 15.6

For each of the following pairs of roots, obtain a corresponding equation in the form $ax^2 + bx + c = 0$, where a, b and c are integers:

1. (a) $(-2, 2)$ (b) $\left(\frac{1}{2}, \frac{1}{3}\right)$
2. (a) $\left(\frac{1}{4}, -3\right)$ (b) $(-3, -5)$
3. (a) $(0.5, 0.75)$ (b) $(0, -3)$
4. (a) $(-3, -3)$ (b) $\left(-2\frac{1}{2}, 3\frac{1}{2}\right)$
5. (a) (p, q) (b) $\left(\frac{1}{p}, \frac{1}{q}\right)$

From Given Situations

Example 7

A rectangular room is 4 metres longer than it is wide. If its area is $12\ m^2$, find its dimensions.

Solution

Let the width of the room be x metres. Its length is then $x + 4$ metres.

The area of the room is $x(x + 4)\ m^2$

Therefore, $x(x + 4) = 12$

$x^2 + 4x = 12$
$\therefore x^2 + 4x - 12 = 0$
$(x + 6)(x - 2) = 0$
Either $x + 6 = 0$ or $x - 2 = 0$
Therefore, $x = -6$ or $x = 2$
One of the answers, $x = -6$ is ignored because the length cannot be negative.
The width of the room is therefore 2 m and the length of the room is;
$x + 4 = 2 + 4$
$= 6$ m

Exercise 15.7

1. The square of a number is 4 more than three times the number. Find the number.
2. Given that the lengths of the three sides of right-angled triangle are x, x + 1 and x + 2 units, find the value of x.
3. A triangle ABC has a base of (x + 3) cm and a height of x cm. If its area is 5 cm^2, calculate the length of its base.
4. After reducing by an equal amount the length and width of a rectangle which were originally 8 cm and 5 cm respectively, the area changes from 40 cm^2 to 18 cm^2. Find the dimensions of the new rectangle.
5. A number of people bought a 300-hectare farm which they shared equally. If the number of hectares per person was 5 less than their number, find the number of people.
6. Figure 15.1 shows two concentric circles. If the ratio of their radii is 1 : 2 and the area of the shaded region is 75 square units, calculate the area of the larger circle.

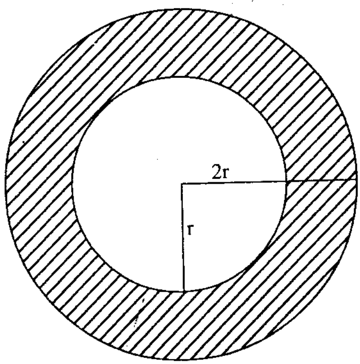


Fig. 15.1

7. A picture measuring 5 cm by 8 cm is mounted on a frame so as to leave a uniform margin of width x cm all round. If the area of the margin is 30 cm^2, find x.
8. A room which measures 10 metres by 5 metres is to be carpeted so as to leave a uniform band of 1 metre all round its walls. Calculate the cost of carpeting this room if one square metre of carpet material costs sh. 300 and labour cost is 15% of the cost of the carpet.
9. Figure 15.2 shows a rectangular lawn of dimensions 6x metres by x metres. In the centre is a rectangular flower garden of length (x + 4) m and width (x – 1) m. If the area of the shaded region is 40 m^2, calculate the area of the flower garden.

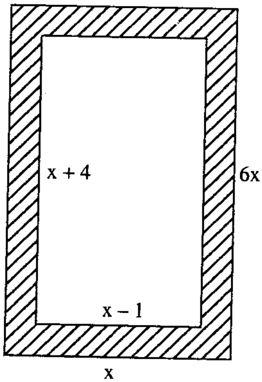


Fig. 15.2

10. There were 240 exercise books to be given to Form 2W. Thirty-four students in the class did not receive the books for various reasons. If the books were then shared equally among the remaining members of the class, the number obtained by each student was equal to the number of students in the class. Find the number of students in the class.
11. Members of a group decided to raise K£100 towards a charity. Five of them were unable to contribute. The rest had therefore to pay K£1 more each to realise the same amount. How many members were in the group originally?
12. ABC is an isosceles triangle in which AB = AC. The size of angles ABC and ACB are $(3x^2 - 2x + 4)^0$ and $(9x - 6)^0$ respectively. Calculate the two possible sets of the three angles of triangle ABC.

Chapter Sixteen

LINEAR INEQUALITIES

16.1: Inequality Symbols

In this section, we need to remind ourselves of the following symbols:

$>$ means greater than, e.g., $5 > 2$.

$<$ means less than, e.g., $4 < 7$.

$\geq$ means greater than or equal to.

$\leq$ means less than or equal to.

Statements connected by these symbols are called **inequalities.**

Simple Statements

Whereas $x = 2$ represents a specific number, $x > 2$ does not. It represents all numbers to the right of 2, as illustrated on the number line below:

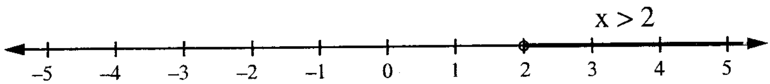


Fig. 16.1

In figure 16.1, the empty circle around the number 2 means that 2 is not included in the list of numbers to the right of it.

The expression $x \geq 2$ on the other hand means that 2 is included in the list of numbers to the right of it. This information can be illustrated on the number line as in figure 16.2:

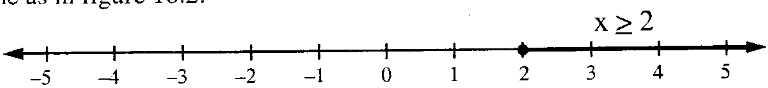


Fig. 16.2

Note that the circle around 2 is now shaded to show that 2 is also included.

Exercise 16.1

Illustrate each of the following inequalities on the number line:

1. (a) $x < 7$ (b) $x > -3$ (c) $x \leq 0$ (d) $x \leq -5$
2. (a) $x < -10$ (b) $x < -4$ (c) $x \geq -6$ (d) $x < 2.5$
3. (a) $x \leq -\frac{1}{2}$ (b) $x \leq -2.3$

Compound Statements

Sometimes we may be required to represent on a number line all values of x which satisfy two conditions simultaneously, e.g., $x > 2$ and $x \leq 8$.

Combined into a one compound statement, the two become;

$2 < x \leq 8$

This is illustrated in figure 16.3. The compound statement $2 < x \leq 8$ represents an interval with 2 and 8 as its endpoints, but 2 is not included.

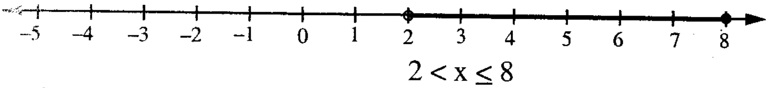


Fig. 16.3

Example 1

Write each of the following pairs of simple statements into compound statements and illustrate them on a number line:

(a) $x > -4, x \leq 2$

(b) $x < 0.5, x > 0$

Solution

(i) $-4 < x \leq 2$

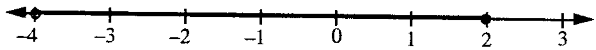


Fig. 16.4

(ii) $0 < x < 0.5$

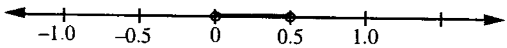


Fig. 16.5

Exercise 16.2

1. Write each of the following pairs of simple statements as a compound statement:
   (a) $x > 2, x < 5$ (b) $x \geq 3, x < 6$
   (c) $x \geq 1, x \leq 7$ (d) $x > -4, x \leq 0$
   (e) $x \geq -3, x \leq -1$
2. Write each of the following pairs of simple statements into a compound statement and illustrate the answer on a number line:
   (a) $x > 2, x < 6$ (b) $x > -4, x \leq 6$ (c) $x \leq 0.5, x \geq 0.125$
   (d) $x < \frac{1}{6}, x > \frac{1}{8}$ (e) $x \leq 4, x \geq 6$ (f) $x \geq 2, x \leq -3$
3. Illustrate the following compound statements on the number line:
   (a) $-1 \leq x < 4$ (b) $-2 < x < 0$ (c) $3 \leq x < 7$ (d) $-5 < x < 5$

(e) $-5 \leq x \leq 3$ (f) $-2 < x < 2$ (g) $-4 < x \leq -1$ (h) $-9 \leq x \leq 15$

(i) $0 \leq x \leq 10$ (j) $4 > x \geq 1$ (k) $\frac{1}{5} < x < \frac{1}{3}$ (l) $2\frac{1}{2} \leq x < 3\frac{1}{2}$

(m) $-0.75 < x \leq 0.75$ (n) $-15 \leq x \leq -3$ (p) $-\frac{1}{2} \leq x \leq 5\frac{1}{2}$

16.2: Solution of Simple Inequalities

Example 2

Solve the inequality $x - 1 > 2$

Solution

Adding 1 to both sides gives;

$x - 1 + 1 > 2 + 1$

Therefore, $x > 3$

It can be seen that if we substitute any number greater than 3 in the inequality, the inequality remains true. In any given inequality you may:

(i) add the same number to both sides, or,

(ii) subtract the same number from both sides.

Example 3

Solve the inequality $x + 3 < 8$.

Solution

Subtracting 3 from both sides gives;

$x + 3 - 3 < 8 - 3$

$x < 5$

Example 4

Solve the inequality $2x + 3 \leq 5$

Solution

Subtracting 3 from both sides gives;

$2x + 3 - 3 \leq 5 - 3$

$2x \leq 2$

Dividing both sides by 2 gives;

$\frac{2x}{2} \leq \frac{2}{2}$

Therefore, $x \leq 1$

We check our answer by substituting back in the inequality any number less than or equal to 1

Example 5

Solve the inequality $\frac{1}{3}x - 2 \geq 4$

Solution

Adding 2 to both sides;

$\frac{1}{3}x - 2 + 2 \geq 4 + 2$

$\frac{1}{3}x \geq 6$

Multiplying both sides by 3;

$\frac{x}{3} \times 3 \geq 6 \times 3$

$x \geq 18$

Multiplication and Division by a Negative Number

Consider the simple statement $5 > 2$.

Let us multiply both sides by -3;

L.H.S. $= -15$, R.H. S. $= -6$. But -6 is greater than -15.

Therefore, the correct statement after multiplication is $-15 < -6$.

The sense of the inequality sign has been reversed.

Generally:

(i) multiplying or dividing both sides of an inequality by a positive number leaves the inequality sign unchanged.

(ii) multiplying or dividing both sides of an inequality by a negative number reverses the sense of the inequality sign.

Example 6

Solve the inequality $1 - 3x < 4$

Solution

Subtracting 1 from both sides gives;

$1 - 3x - 1 < 4 - 1$

$-3x < 3$

Dividing both sides by -3 gives;

$\frac{-3x}{-3} > \frac{3}{-3}$

(Note that the sign is reversed)

Therefore, $x > -1$

It is advisable to check your answer by substituting back in the original inequality.

Exercise 16.3

Solve each of the following inequalities and represent your solutions on a number line.

1. (a) $2x + 4 > 10$ (b) $3x - 5 < 2$
2. (a) $5x + 3 > 4$ (b) $3x - 4 \leq -13$
3. (a) $3x - 7 \geq 5$ (b) $1 - 4x \geq 9$
4. (a) $6 - \frac{1}{2}x > 12$ (b) $3 - 2x < 17$
5. (a) $\frac{1}{3} - 2x \leq -8\frac{1}{3}$ (b) $3(1 - x) + 4(x + 3) \geq 30$
6. (a) $2x + 3 < -1$ (b) $-3x - 4 \geq 2$
7. (a) $-4 - \frac{2}{3}x \leq 0$ (b) $\frac{x - 7}{-49} \leq -\frac{1}{7}$

16.3: Simultaneous Inequalities

Example 7

Solve the following pair of simultaneous inequalities:

$3x - 1 > -4$

$2x + 1 \leq 7$

Solution

Solving the first inequality;

$3x - 1 > -4$

$3x > -3$

$x > -1$

Solving the second inequality;

$2x + 1 \leq 7$

$2x \leq 6$

$x \leq 3$

The solution which satisfies both inequalities is $-1 < x \leq 3$. This is shown on the number line in figure 16.6.

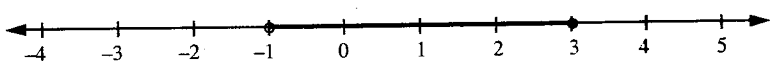


Fig. 16.6

Simultaneous inequalities can as well be written as a compound statement. For instance, $x - 4 < 4x < 4$ should be interpreted as;

(i) $x - 4 < 4x$

(ii) $4x < 4$

Solving the first inequality,

$x < 4x + 4$

$-3x < 4$

$x > -\frac{4}{3}$

Solving the second inequality;

$4x < 4$

$x < \frac{4}{4}$

$x < 1$

The solution which satisfies both inequalities is, therefore;

$-\frac{4}{3} < x < 1$

Exercise 16.4

Solve each of the following simultaneous inequalities and illustrate your answers on a number line:

1. (a) $x + 3 > 5$, $x - 4 < 4$ (b) $x + 10 \geq 6$, $x - 2 \leq 3$

2. (a) $-\frac{1}{2}x - 2 \leq 1$, $-3x - 9 > -6$ (b) $-5x + 7 < 12$, $\frac{1}{3}x + 2 \leq 5$

3. (a) $-7x - 1 < 6$, $\frac{x}{3} + 1 < \frac{4}{3}$ (b) $3x - \frac{1}{2} > 4$, $x - \frac{1}{5} < \frac{2}{5}x + 1$

4. (a) $\frac{x}{5} + \frac{1}{3} < 1$, $x - \frac{4}{5} > \frac{1}{8}x$ (b) $5 \leq 3x + 2$, $3x - 14 < -2$

5 (a) $\frac{x}{3} + \frac{2}{3} > 2$, $-\frac{1}{2}x + 1 < 2$ (b) $\frac{x - 3}{-3} > 1$, $3x + 1 < -17$

6. (a) $\frac{x + 2}{2} < 5$, $\frac{x + 6}{3} < 4$ (b) $9 - 2x \leq 3$, $1 < 16 - x$

7. (a) $\frac{1}{2} - \frac{1}{4}x \leq x \leq 2$ (b) $12 - x \geq 5 \leq 2x - 2$

8. (a) $-4x < 6 \leq 80x$ (b) $3x - 2 \geq -4 < -1 - 2x$

9. (a) $6x - 13 \leq 17 < 8x - 7$ (b) $2x + 3 > 5x - 3 > -8$

16.4: Graphical Representation of Inequalities

Figure 16.7 shows the graph of $x = 2$:

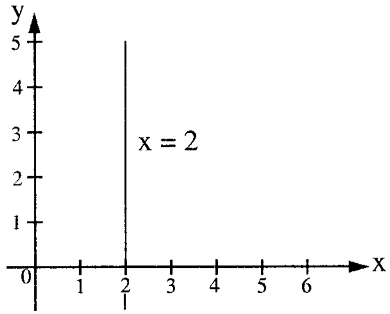


Fig. 16.7

Give the co-ordinates of any four points:

(i) on the line $x = 2$.

(ii) to the right of the line $x = 2$.

(iii) to the left of the line $x = 2$.

You may have noticed that all points lying in the region to the right of $x = 2$ satisfy the inequality $x > 2$.

Write down an inequality for those points lying in the region to the left of $x = 2$.

Points which either lie on the line $x = 2$ or to the right of it are represented by the inequality $x \geq 2$. Figure 16.8 shows the unshaded region satisfied by the inequality $x \geq 2$.

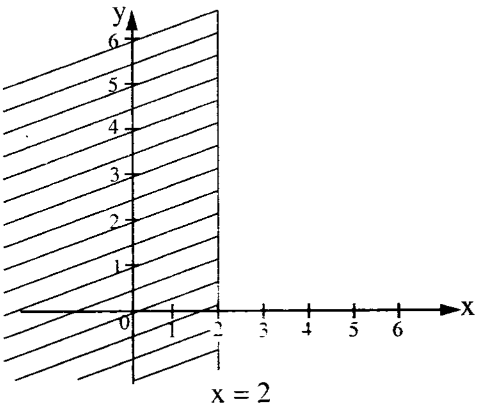


Fig. 16.8

Note:

(i) Always shade the region which does not satisfy the inequality (unwanted region).

(ii) The line $x = 2$ has been drawn continuous (unbroken) because it forms part of the region (all points on the line satisfy the inequality $x \geq 2$).

The region $x > 2$ is shown in figure 16.9..

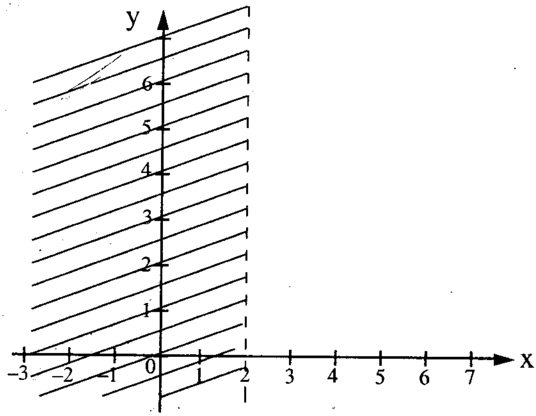


Fig. 16.9

Note:

In this case, the line $x = 2$ is broken. This is because points on the line do not form part of the region (they do not satisfy the inequality $x > 2$).

Exercise 16.5

Show the regions that satisfy each of the following inequalities on a squared paper:

1. (a) $x \leq 4$ (b) $x > -2$
2. (a) $x < -1$ (b) $y \leq 3$
3. (a) $y \geq -4$ (b) $y \leq 0$
4. (a) $y + 2 < -5$ (b) $x + 2 \geq -1$
5. (a) $3 - x > 7$ (b) $\frac{x + 1}{3} < 6$
6. (a) $\frac{y - 1}{4} \geq 5$ (b) $\frac{y + 3}{5} \leq 2$
7. (a) $-2 \leq x < 4$ (b) $-4 < y \leq 3$
8. (a) $-2 \leq \frac{1}{2}x < 7$ (b) $4 < y < 6$

9. (a) $-1 \le 3x - 1 < 5$ (b) $\frac{x-3}{4} > \frac{x+5}{2}$

10. (a) $\frac{1}{3} - \frac{1}{5}x < \frac{1}{2}x + 1$ (b) $\frac{2}{3}x - 7 + \frac{1}{5}x \ge -1$

11. (a) $\frac{1}{2}x + \frac{1}{2} > \frac{1}{4}$ (b) $x^2 - 4x \ge x(x-1) - 18$

12. (a) $\frac{1}{7}x - \frac{1}{3} < \frac{1}{5} + \frac{x}{3}$ (b) $x - 2 \ge 4 + 3x$

16.5: Linear Inequalities in Two Unknowns

In the previous section, we drew regions representing inequalities of the form $x \ge d$ or $x \le e$, where d and e are constants. In this section, we will show how to represent inequalities of the form $ax + by \ge c$ graphically (where a, b and c are constants).

For example, the region represented by $x + y \ge 8$ is shown in figure 16.10. The boundary is the line $x + y = 8$.

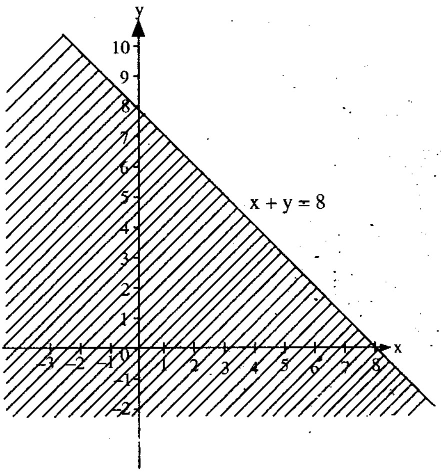


Fig. 16.10

If we pick any point above the line, e.g., (10, 10), then $x + y = 10 + 10 = 20$. Since $20 > 8$, the point (10, 10) lies in the wanted region. On the other hand, if we pick a point below the line, such as (2, 3), we have;

$x + y = 2 + 3 = 5$. Since $5 < 8$, this point does not satisfy the inequality. The region below the line therefore does not satisfy the inequality and that is why it is shaded. Show on the graph the region $x + y < 8$.

Given a region satisfied by an inequality, the inequality can be found. For example, consider the region below:

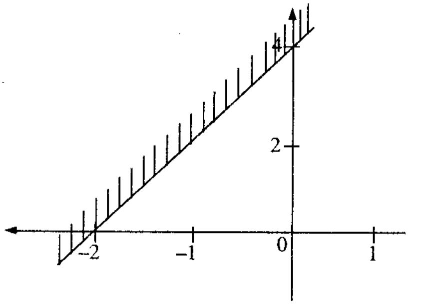


Fig. 16.11

The equation of the line is $y = 2x + 4$.

Consider a point in the wanted region such as the origin (0, 0). The x co-ordinate and y co-ordinate are both zero. Substituting these values in the equation $y = 2x + 4$, we get zero on the left hand side and four on the right. Since $0 < 4$ and the line $y = 2x + 4$ is continuous, the required inequality is $y \le 2x + 4$.

Exercise 16.6

Graph each of the following inequalities:

1. (a) $2x + y > 3$ (b) $x - y < 4$
2. (a) $3x + 2y > 12$ (b) $3y + x \le -5$
3. (a) $y + 4x < 3$ (b) $y - \frac{1}{2}x \ge 1$
4. (a) $2x > y + 4$ (b) $\frac{1}{5}x + \frac{1}{3}y \le \frac{1}{4}$
5. (a) $2y - 3x - 5 \le 0$ (b) $\frac{1}{6}x + \frac{1}{3}y \le -1$
6. (a) $2y - 5x \ge 7$ (b) $\frac{1}{2}x - y \ge 1$
7. (a) $3\left(\frac{1}{2}x - \frac{1}{y}y\right) \le -1$ (b) $3y - 8 \ge -6x$
8. (a) $1 < x + y < 8$ (b) $2y \ge 0.75 - x$
9. $0.5x + 0.3y > -1$

10. Determine the inequalities which satisfy the following unshaded regions:

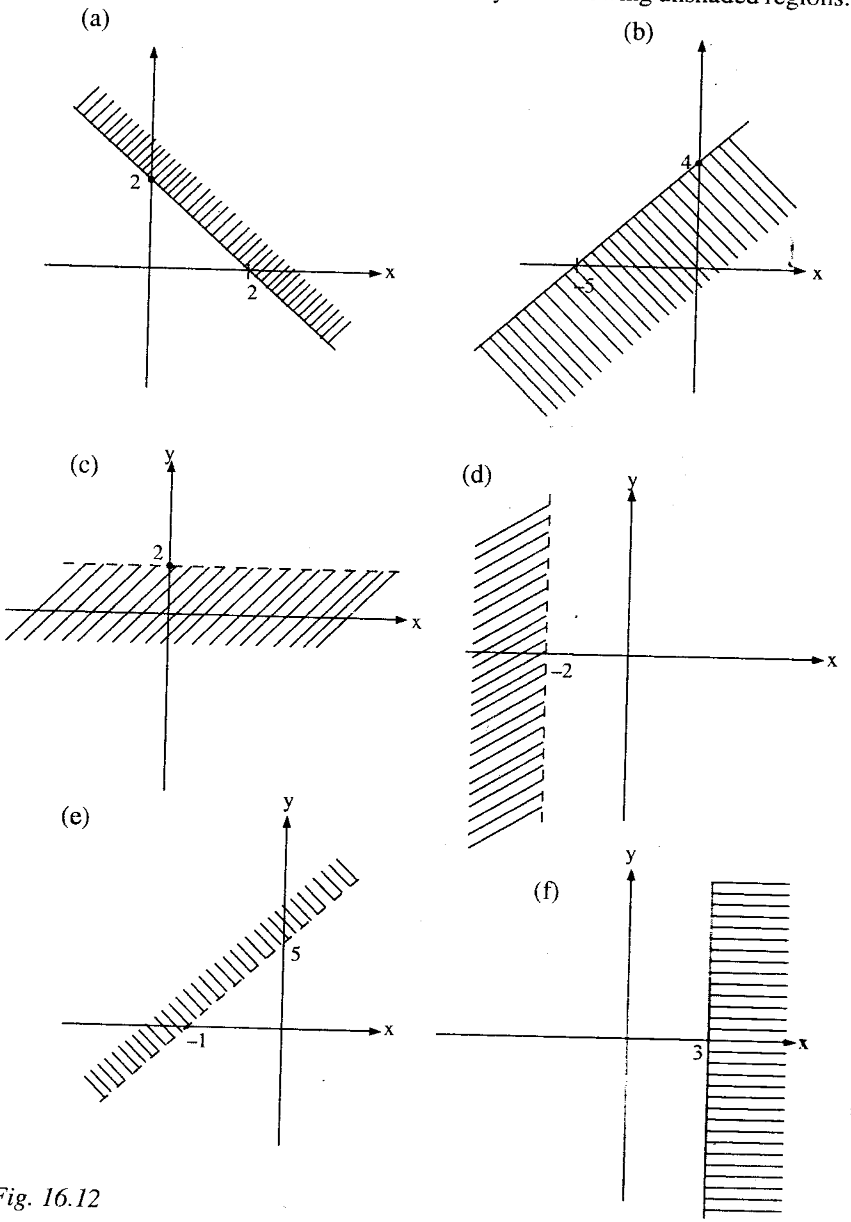


Fig. 16.12

16.6: Intersecting Regions

So far we have been able to identify regions which satisfy only one inequality. Situations sometimes arise in which we are required to identify regions which satisfy more than one inequality simultaneously. Consider the following examples.

Example 8

Draw a region which satisfies both the inequalities $x + y \leq 8$ and $x > 2$.

Solution

The solution is shown in figure 16.13. The required region is unshaded.

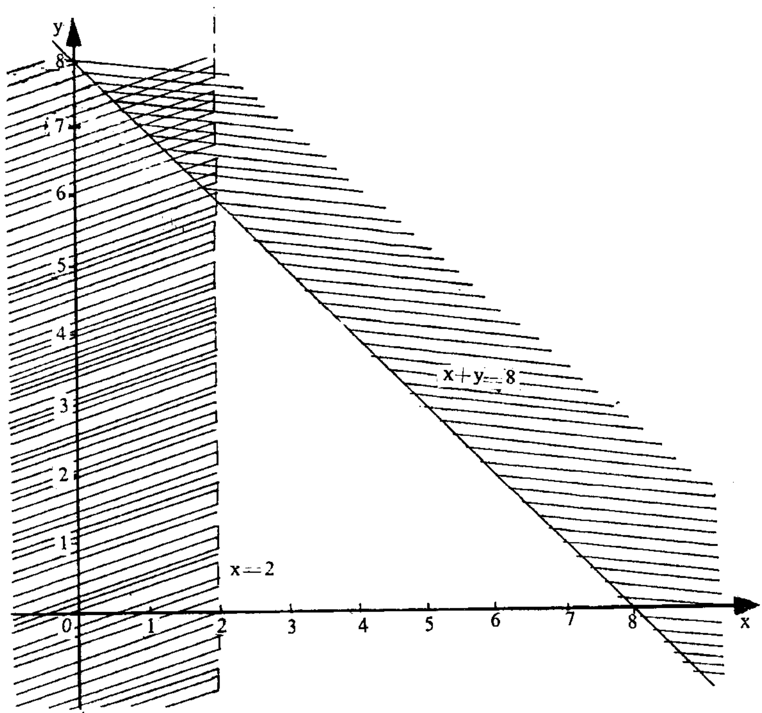


Fig. 16.13

Example 9

Show the region represented by $2x + y > 3$, $x - y \geq 4$ and $y \leq 3$.

Solution

The solution is shown in figure 16.14.

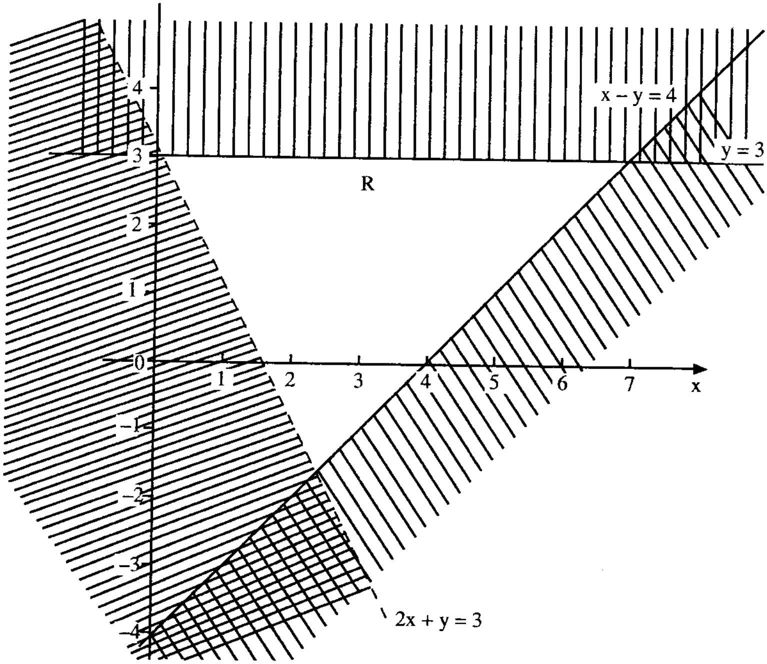


Fig. 16.14

Note that the required region (R) is unshaded.

Exercise 16.6

In each of the questions 1-5, draw the regions which satisfy all the inequalities.

1. (a) $x + y \geq 0$, $x < 2$, $y > 0$ (b) $2x + y \geq 6$, $x < 3$, $y < 6$

2. (a) $4x - 3y \leq 12$, $x > 0$, $y > 0$ (b) $4x - 3y < 12$, $y \geq 0$, $y \leq 6$

3. (a) $y + 2x \leq 5$
$y < 4x + 12$

(b) $y - x < 0$
$x \leq 5$
$y \geq 0$

4. (a) $2x + y > 4$
$6y + 2y \leq 12$

(b) $x + y \leq 6$
$y > 4$
$x + 3 > 0$

5. (a) $5x + 3y \geq 15$
$6y + 5x \leq 30$
$y \geq 0$

(b) $0 \leq y < 3$
$0 \leq x < 4$

6. Find the areas of the required regions in questions 5 (a) and (b).

7. Write down the co-ordinates of any four points with integral values which lie in the region obtained in question 4 (a).

8. The vertices of the unshaded triangular region in figure 16.15 are O(0,0), A(8, 0) and B(8, 8). Write down the inequalities which are satisfied by the region.

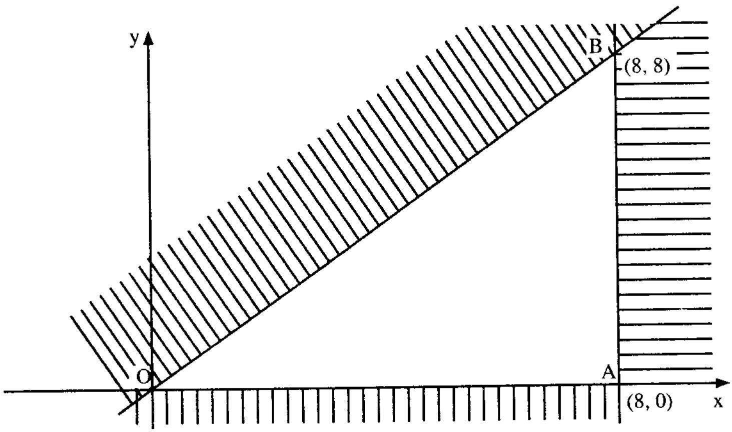


Fig. 16.15

9. Figure 16.16 shows a square ABCD with vertices A(5, 0), B(0, 5) C(–5, 0) and D.

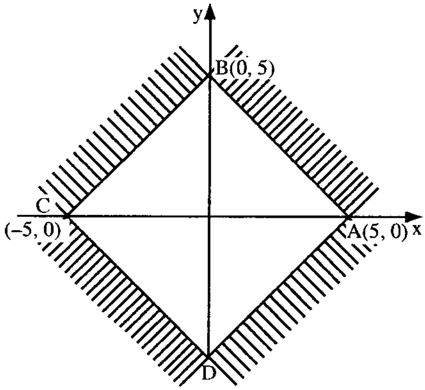


Fig. 16.16

(a) Determine the co-ordinates of point D.
(b) Write down the equations of lines AB, CB, CD and AD.
(c) Write down the inequalities which determine the square.

10. Figure 16.17 shows the vertices of a region bounded by lines AB, BC, CD and DA.

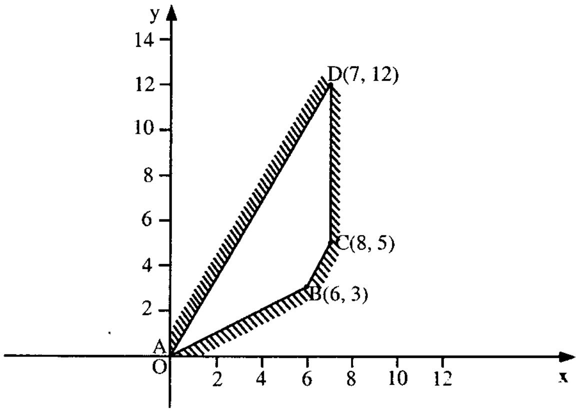


Fig. 16.17

Write down the inequalities that determine the region ABCD.

11. Given the inequalities $y - 2x \leq 4$ and $x + y \leq 1$, find two other inequalities which, together with these two, will complete a parallelogram with (−2, 0) and (1, 0) as two of its vertices.

Chapter Seventeen

LINEAR MOTION

17.1: Definitions

The distance between two points is the length of the path joining them. The SI unit of distance is the metre. Displacement is distance in a specified direction.

Speed is the rate of change of distance with time. The SI unit of speed is m/s, though speed is commonly given in km/h. Velocity is speed in a specified direction or the rate of change of displacement with time.

Acceleration on the other hand is the rate of change of velocity with time. The SI unit of acceleration is m/s^2.

17.2: Speed

Over a time interval, the speed normally varies and what we often use is the average speed.

$$\text{Average speed} = \frac{\text{distance covered}}{\text{time taken}}$$

If a motorist covers a distance of 180 km in 3 hours, his average speed is $\frac{180}{3}$ km/h = 60 km/h.

Example 1

A man walks for 40 minutes at 60 km/h. He then travels for two hours in a minibus at 80 km/h. Finally, he travels by bus for one hour at 60 km/h. Find his speed for the whole journey.

Solution

$$\text{Average speed} = \frac{\text{distance covered}}{\text{time taken}}$$

$$\text{Total distance} = \left(\tfrac{40}{60} \times 60 \text{ km}\right) + (2 \times 80) \text{ km} + (1 \times 60) \text{ km}$$

$$= 260 \text{ km}$$

$$\text{Total time taken} = \tfrac{4}{6} + 2 + 1$$

$$= 3\tfrac{2}{3} \text{ hrs}$$

$$\text{Average speed} = \frac{260}{3\frac{2}{3}}$$

$$= \frac{260 \times 3}{11}$$

$$= 70.9 \text{ km/h}$$

Exercise 17.1

1. Juma cycled for $2\frac{1}{2}$ hours to Mula trading centre, 30 km away from his home. He then took his business van and drove at an average speed of 80 km/h to a town 180 km away and back to Mula. He finally cycled home at an average speed of 15 km/h. Find the average speed for the whole journey.
2. A motorist travelled for 2 hours at a speed of 80 km/h before his vehicle broke down. It took him half an hour to repair the vehicle. He then continued with his journey for $1\frac{1}{2}$ hours at a speed of 60 km/h. What was his average speed?
3. John travelled from town A to town B, a distance of 30 km. He cycled for $1\frac{1}{2}$ hours at a speed of 16 km/h and then walked the remaining distance at a speed of 3 km/h. Find his average speed for the journey.
4. Anne takes two hours to walk from home to her place of work, a distance of 8 km. On a certain day, after walking for 30 minutes, she stopped for ten minutes to talk to a friend. At what average speed should she walk to reach on time?
5. A motorist drove for 1 hour at 100 km/h. She then travelled for $1\frac{1}{2}$ hours at a different speed. If the average speed for the whole journey was 88 km/h, what was the average speed for the latter part of the journey?
6. A commuter train moves from station A to station D via stations B and C in that order. The distance from A to C via B is 70 km and that from B to D via C is 88 km. Between the stations A and B, the train travels at an average speed of 48 km/h and takes 15 minutes. Between the stations C and D, the average speed of the train is 45 km/h. Find:
   (a) this distance from B to C.
   (b) the time taken between C and D.
   (c) If the train halts at B for 3 minutes and at C for 4 minutes and its average speed for the whole journey is 50 km/ h, find its average speed between B and C.

17.3: Velocity and Acceleration

Consider two cars moving in opposite directions, each at a speed of 65 km/h. Their velocities are different because they are travelling in different directions. For motion under constant acceleration;

$$\text{Average velocity} = \frac{\text{initial velocity} + \text{final velocity}}{2}$$

Consider a car moving in a given direction under constant acceleration. If its velocity at a certain time is 75 km/h and 10 seconds later it is 90 km/h, then;

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time taken}}$$

$$= \frac{(90 - 75)\text{ km/h}}{10\text{ s}}$$

$$= \frac{(90 - 75) \times 1\,000}{10 \times 60 \times 60}\text{ m/s}^2$$

$$= \tfrac{5}{12}\text{ m/s}^2$$

Now, consider a car moving with a velocity of 50 km/h then the brakes are applied so that it stops after 20 seconds. In this case, the final velocity is 0 km/h and initial velocity $= 50$ km/h.

$$\text{Therefore, acceleration} = \frac{(0 - 50) \times 1\,000}{20 \times 60 \times 60}\text{ m/s}^2$$

$$= -\tfrac{25}{36}\text{ m/s}^2$$

The acceleration is negative.

Negative acceleration is usually referred to as deceleration or retardation. Here, the retardation is $\frac{25}{36}$ m/s^2.

Exercise 17.2

1. The initial velocity of a car is 10 m/s. The velocity of the car after 4 seconds is 30 m/s. Find its acceleration.
2. A bus accelerates from a velocity of 12 m/s to a velocity of 25 m/s. Find the average velocity during this interval.
3. A car moves with a constant acceleration of 8 m/s^2 for 5 seconds. If the final velocity is 40 m/s, find the initial velocity.
4. A train driver moving at 40 km/h applies brakes so that there is a constant retardation of 0.5 m/s^2. Find the time taken before the train stops.

5. The velocity of a particle increases uniformly from 27 m/s to 57 m/s in one minute. Find the acceleration.
6. The driver of a minibus which is moving at 80 km/h notices a road block a short distance ahead. He applies brakes immediately so that the minibus moves with constant deceleration for 5 seconds before coming to rest at the road block. Find:
   (a) the average velocity of the minibus during the 5 seconds.
   (b) the deceleration period.

17.4: Distance-Time Graph

When distance is plotted against time, a distance-time graph is obtained.

Example 2

Table 17.1 shows the distance covered by a motorist from Limuru to Kisumu:

Table 17.1

Time	9.00 a.m.	10.00 a.m.	11.00 a.m.	11.30 a.m.	12.00 noon	1.00 p.m.
Distance (km)	0	80	160	160	210	310

(a) Draw the distance-time graph.

(b) Use the graph to answer the following questions:

   (i) How far was the motorist from Limuru at 10.30 a.m?

   (ii) What was the average speed during the first part of the journey?

(c) What was the average speed for the whole journey?

Solution

(a)

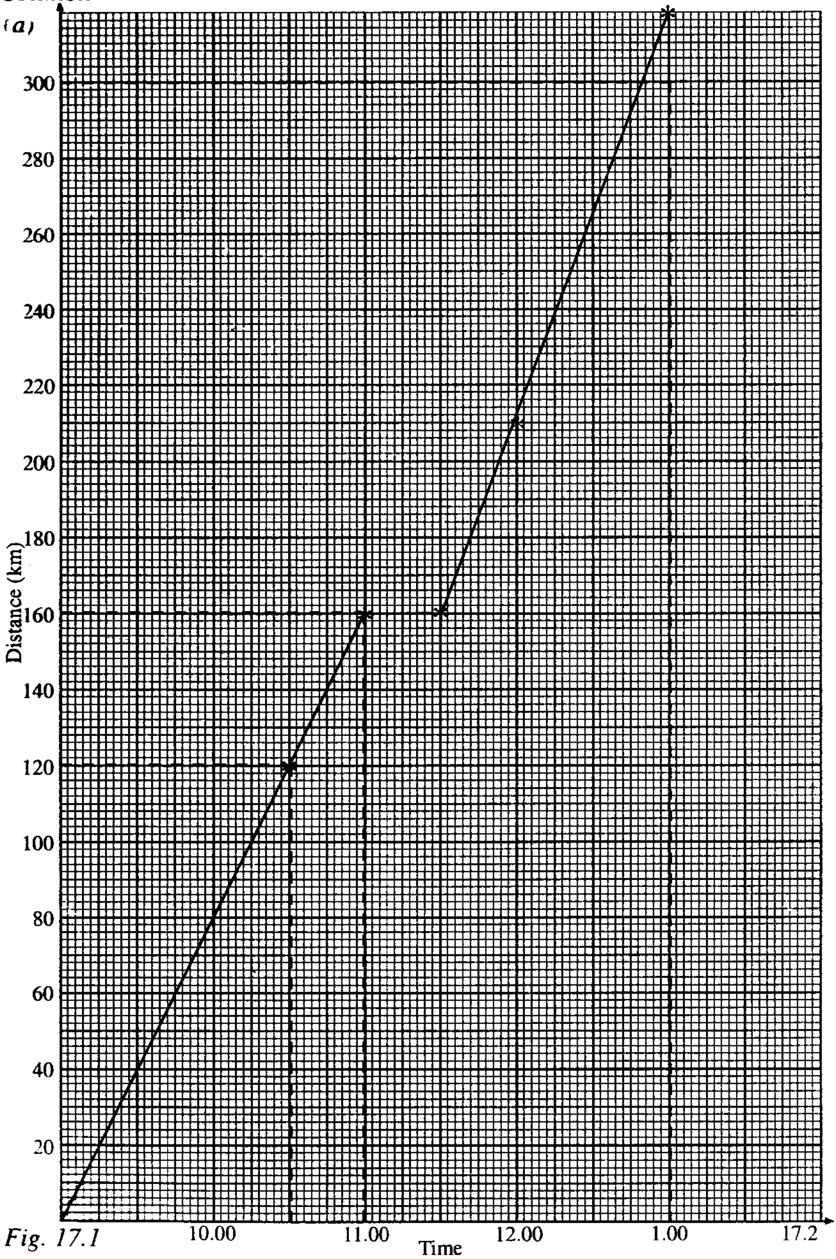


Fig. 17.1

17.2

(b) (i) 120 km (ii) $\frac{160 \text{ km}}{2 \text{ h}} = 80$ km/h

(c) Average speed = $\frac{\text{total distance}}{\text{total time}} = \frac{310 \text{ km}}{4 \text{ h}} = 77.5$ km/h

Exercise 17.3

1. Figure 17.2 shows a travel graph for a motorist who starts his journey at 12 noon and drives to a town 450 kilometres away:

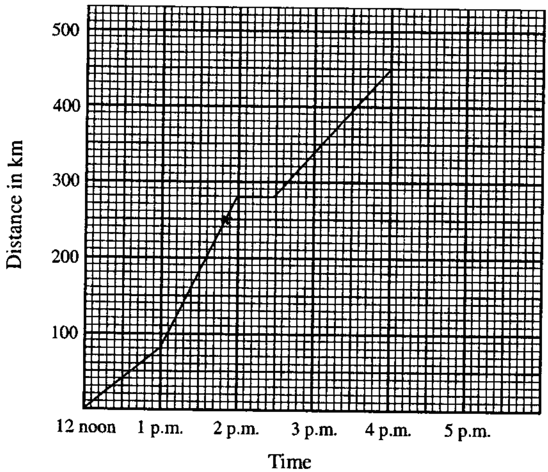


Fig. 17.2

Use the figure to answer the questions below:

(a) Find the average speed of the motorist:
  (i) between 12 noon and 1 p.m. (ii) between 1 p.m. and 2 p.m.
  (iii) between 1 p.m. and 3 p.m. (iv) for the whole journey.
  (v) for the first 200 kilometres. (vi) for the last 200 kilometres.

(b) Find the distance covered between:
  (i) 12 noon and 2 p.m. (ii) 1.30 p.m. and 3.30 p.m.

(c) Between what times was the average speed greatest?

2. A man leaves home at 9.00 a.m. and walks to a bus stop 6 km away at an average speed of 4 km/h. He then waits at the bus stop for 25 minutes before boarding a bus to a town 105 km away. The bus travels at an average speed of 60 km/h. Draw a distance time-graph for the journey and use it to answer the following questions:

(a) At what time was the man 100 km from home?

(b) How far away from home was he at 10.15 a.m.?

3. A motorist left Eldoret at 11.00 a.m. for Nairobi, a distance of 320 km away. She took $1\frac{1}{2}$ hours to reach Nakuru half-way between Eldoret and

Nairobi where it took her 20 minutes to fuel the car. She then set off for Nairobi, but stopped after 25 minutes at Keekopey, 30 km from Nakuru, for 30 minutes. She then proceeded with the journey to Nairobi and arrived at 5.00 p.m.

(a) Draw the distance-time graph to show the journey.

(b) Use the graph to find:

(i) the average speed for the whole journey.

(ii) the average speed of the last part of the journey.

17.5: Velocity-Time Graph

When velocity is plotted against time, a velocity time graph is obtained.

Consider a particle which starts from rest and moves in a straight line. For the first four seconds, the acceleration is 10 m/s^2. The velocity then remains constant for the next two seconds before the particle undergoes a deceleration of 8 m/s^2 and eventually comes to rest. This information is represented in table 17.2.

Table 17.2

Time in seconds	0	1	2	3	4	5	6	7	8	9	10	11
Velocity in m/s	0	10	20	30	40	40	40	32	24	16	8	0

The velocity-time graph is given in figure 17.3.

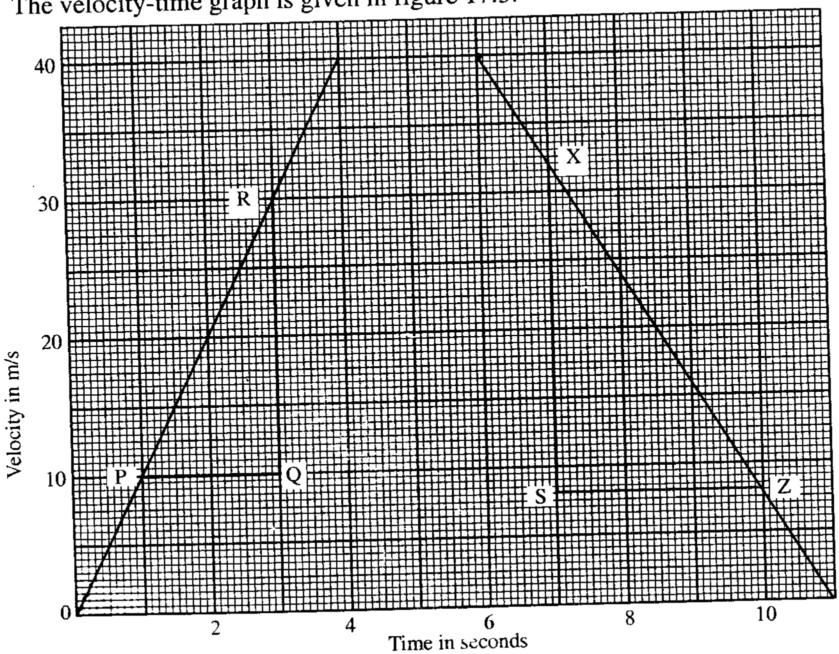


Fig. 17.3

Note that in the figure, the ratio;

$$\frac{\text{change in velocity}}{\text{corresponding change in time}}$$ = acceleration So long as acceleration is constant in the time interval considered. For example;

$$\frac{QR}{PQ} = \frac{(30 - 10)\ \text{m/s}}{(3 - 1)\ \text{s}}$$

$$= 10\ \text{m/s}^2$$

This is the acceleration for the first four minutes.
Similarly;

$$\frac{XY}{YZ} = \frac{(8 - 32)\ \text{m/s}}{(10 - 7)\ \text{s}}$$

$$= -8\ \text{m/s}^2$$

This is the retardation for the last five minutes.
In general, the slope of the graph in a velocity-time graph represents the acceleration.

To see how distance covered can be obtained from a velocity-time graph, let us now consider a car moving for two hours at a constant velocity of 80 km/h. The velocity-time graph is shown in figure 17.4.

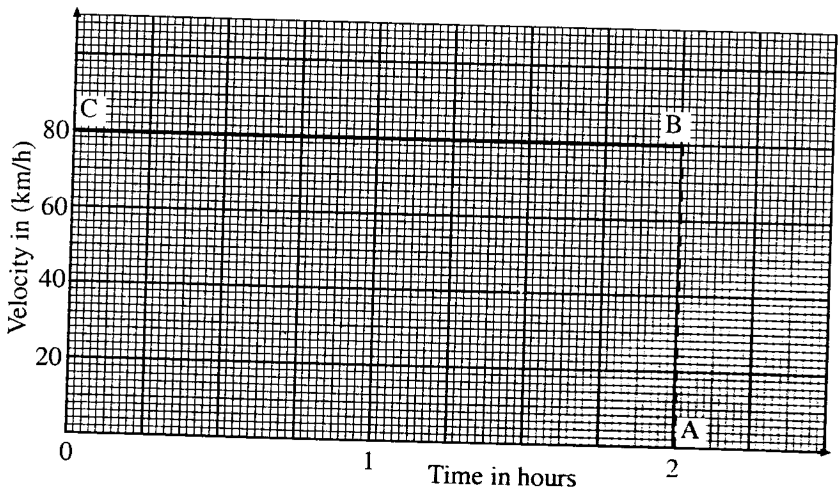


Fig. 17.4

Distance covered in 2 hours is given by;
80 km/h × 2 h = 160 km

It should be noted that this distance is equal to the area of the rectangle OABC.

In general, the area under a velocity-time graph represents the total distance covered. Thus, in figure 17.5, the shaded area represents the total distance covered between the times t_1 and t_2.

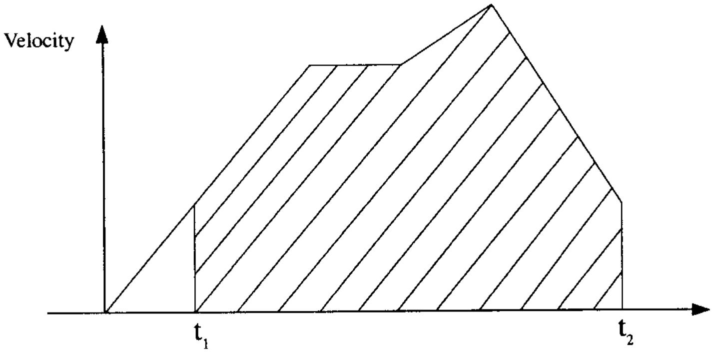


Fig. 17.5

Table 17.3 gives velocity of a train moving with a constant acceleration of 5 m/s^2.

Table 17.3

Time in seconds	0	1	2	3	4	5
Velocity in m/s	0	5	10	15	20	25

Figure 17.6 gives the velocity-time graph.

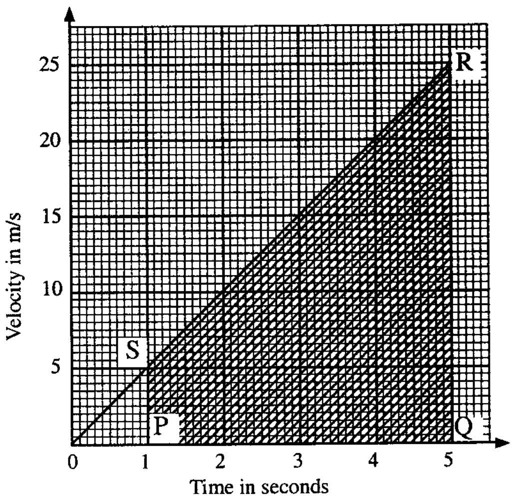


Fig. 17.6

The shaded area, which is a trapezium, gives the distance covered between the first and the fifth seconds (i.e., in 4 seconds).

$$\begin{aligned}\text{Distance} &= \text{area of trapezium}\\ &= \tfrac{1}{2} \times 4(5 + 25)\\ &= 2 \times 30\\ &= 60 \text{ km}\end{aligned}$$

This is equal to the distance covered within 4 seconds at a constant velocity of;

$$\frac{5 \text{ m/s} + 25 \text{ m/s}}{2} = 15 \text{ m/s}$$

Exercise 17.4

1. A car is travelling at 40 m/s. Its brakes are applied and it then decelerates at 8 m/s^2. Use a velocity-time graph to find the distance it travels before stopping.
2. A particle is projected vertically upwards with a velocity of 30 m/s. If the retardation to motion is 10 m/s^2, use a graphical method to find the maximum height reached by the particle.
3. Figure 17.7 shows a velocity-time graph of a particle moving along a West-East straight line, with velocity considered positive eastwards:

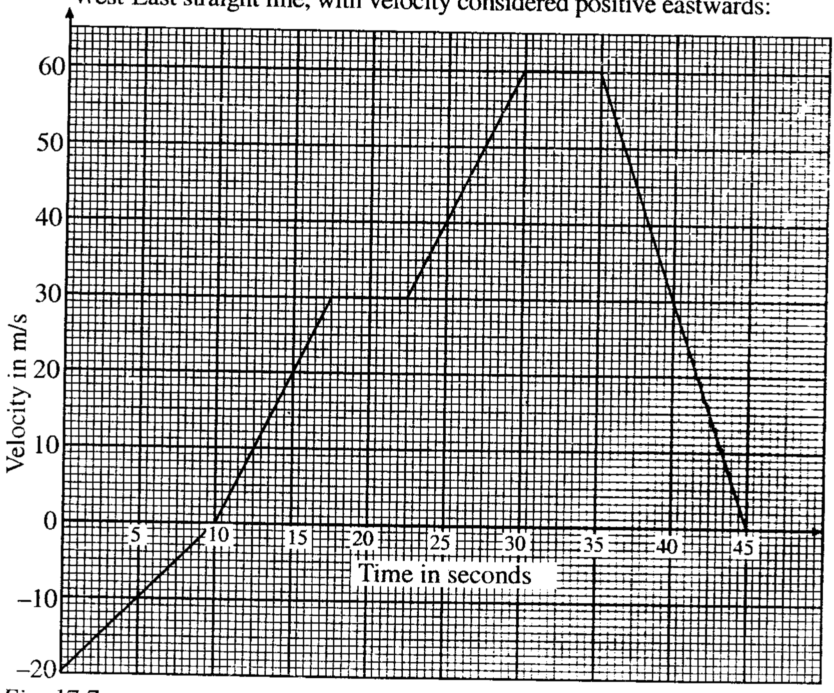


Fig. 17.7

Use the graph to answer the following questions:

(a) Find the velocity of the particle:
   (i) after 5 seconds.
   (ii) after 15 seconds.
   (iii) after 20 seconds.
   (iv) after 40 seconds.

(b) Find the acceleration of the particle:
   (i) during the first 5 seconds.
   (ii) during the 15^{th} second of the motion.
   (iii) between the 18^{th} and the 20^{th} second.
   (iv) between the 37^{th} and the 41^{st} second.

(c) Use area under the graph to find the distance covered by the particle:
   (i) between the 10^{th} and the 15^{th} second.
   (ii) between the 25^{th} and the 35^{th} second.
   (iii) between the 20^{th} and the 25^{th} second.

(d) Find the average velocity:
   (i) during the period from the 10^{th} to 15^{th} second.
   (ii) during the period from 35^{th} to the 40^{th} second.
   (iii) during the period from 30^{th} to the 35^{th} second.

4. A particle moving with a velocity of 25 m/s along a North-South straight line passes through a point P at 11.00 a.m. If the acceleration is 5 m/s^2 southwards:
   (a) find the velocity of the particle (give direction):
      (i) $4\frac{1}{2}$ seconds after 11.00 a.m.
      (ii) 6 seconds after 11 a.m.
      (iii) 2 seconds before 11 a.m.
   (b) find the average velocity of the particle:
      (i) during the first 10 seconds after 11 a.m.
      (ii) during the first 12 seconds after 11 a.m.
   (c) Use the graphical method to find the distance of the particle from P at 20 seconds past 11.00 a.m.

17.6: Relative Speed

Consider two bodies moving in the same direction at different speeds. Their relative speed is the difference between the individual speeds.

On the other hand, the relative speed when two bodies are approaching each other is the sum of their speeds.

Example 2

A van left Nairobi for Kakamega at an average speed of 80 km/h. After half an hour, a car left Nairobi for Kakamega at a speed of 100 km/h.

(a) Find the relative speed of the two vehicles.

(b) How far from Nairobi did the car overtake the van?

Solution

(a) Relative speed = difference between the speeds

$= 100 - 80$

$= 20$ km/h

(b) Distance covered by van in 30 minutes;

Distance $= \frac{30}{60} \times 80$

$= 40$ km

Time taken for car to overtake ~~matatu~~ Van $= \frac{40}{20}$

$= 2$ hours

Distance from Nairobi $= 2 \times 100$

$= 200$ km

Example 3

A truck left Nyeri at 7.00 a.m. for Nairobi at an average speed of 60 km/h. At 8.00 a.m., a bus left Nairobi for Nyeri at a speed of 120 km/h. How far from Nyeri did the vehicles meet if Nyeri is 160 km from Nairobi?

Solution

Distance covered by lorry in 1 hour $= 1 \times 60$

$= 60$ km

Distance between the vehicles at 8.00 a.m. $= (160 - 60)$ km

$= 100$ km

Relative speed $= 60$ km/h $+ 120$ km/h

$= 180$ km/h

Time taken for the vehicles to meet $= \frac{100}{180}$

$= \frac{5}{9}$ hours

Distance from Nyeri $= 60 + \frac{5}{9} \times 60$

$= 60 + 33.3$

$= 93.3$ km

Exercise 17.5

1. A heavy commercial truck left Murang'a at 8.00 a.m. for Nairobi, a distance of 60 km, at a speed of 40 km/h. If a matatu left Murang'a for Nairobi at a speed of 80 km/h, at what time did the matatu overtake the truck? 14 17
started at 8.30am.
2. How long will it take a car 3 metres long moving at 100 km/h to overtake a truck 10 m long moving at 50 km/h if the car is 10 m behind the truck?
3. A motorist left Embu for Nairobi, a distance of 240 km, at 8.00 a.m. and travelled at an average speed of 90 km/h. Another motorist left Nairobi for Embu at 8.30 a.m. and travelled at 100 km/h. Find:
   (a) the time they met.
   (b) how far they met from Nairobi.
4. A cyclist travelling at 15 km/h left Kitale at 7.00 a.m. for Kakamega, a distance of 70 km. Another cyclist left Kakamega for Kitale at 8.00 a.m. If they met after $2\frac{1}{2}$ hours, what was the speed of the second cyclist?
5. A matatu left town A at 7 a.m. and travelled towards a town B at an average speed of 60 km/h. A second matatu left town B at 8 a.m. and travelled towards town A at 60 km/h. If the distance between the two towns is 400 km, find:
   (a) the time at which the two matatus met.
   (b) the distance of the meeting point from town A.

Chapter Eighteen

STATISTICS

18.1: Introduction

Statistics is a branch of mathematics that deals with collection, organisation, representation and interpretation of data. Data is basic information, usually in number form. When data is not organised, it is said to be raw data. Statistics helps us to make decisions as individuals, organisations or government.

Consider the marks obtained by students in a test, which were as follows:
70, 60, 55, 70, 41, 30, 60, 70, 55, 70, 30, 41, 70, 55, 41, 30, 55, 41.
To make this data meaningful, it may be presented using the rank order list, i.e.,
70, 70, 70, 70, 70, 60, 60, 55, 55, 55, 55, 41, 41, 41, 41, 30, 30, 30.

It is now easy to state how many students attained a certain mark at a glance.

Project

Collect as much data as possible concerning situations such as:

(i) masses, in kilogrammes, of students in your school.
(ii) the age, in years, of the students in your class.
(iii) the height, to the nearest centimetre, of the students in your class.
(iv) the marks obtained in a recent test.
(v) the number of children in a number of families.
(vi) the favourite subjects or drinks of students in a class.
(vii) the favourite games of students in a class.
(viii) the shoe sizes worn by the students in a class.
(ix) the times taken in a 100 metres race by students in your class.
(x) the daily temperature recorded in your school or nearest weather station for one year.

18.2: Frequency Distribution Table

The following data represents the shoe sizes worn by 20 form two students:
7, 9, 6, 10, 8, 8, 9, 11, 8, 7, 9, 6, 8, 10, 9, 8, 7, 7. 8, 9.
The data can be represented as in table 18.1. Such a table is referred to as a frequency distribution table.

Table 18.1

Shoe size	Tally	No. of students (frequency)
6	//	2
7	////	4
8	~~////~~ /	6
9	~~////~~	5
10	//	2
11	/	1
		20

Tally

Each stroke represents a quantity. For example, 4 students put on number 7, therefore we have 4 strokes.

Frequency

This is the number of times an item or value occurs. For example, in the raw data, 8 appears 6 times. Therefore, in the table 18.1, the frequency of 8 is 6. Use the same table to identify the frequency of shoe size 6, 9, 10 and 11.

Example 1

The data below shows marks obtained by 20 students in an essay:

9, 5, 5, 4, 5, 3, 5, 11, 6, 3, 6, 8, 9, 6, 13, 8, 8, 13, 5, 10.

Prepare a frequency distribution table:

Solution

Table 18.2

Marks	*Tally*	*Frequency (f)*
3	//	2
4	/	1
5	~~////~~	5
6	///	3
8	///	3
9	//	2
10	/	1
11	/	1
13	//	2
		20

Note:

The marks are arranged from the lowest to the highest.

18.3: Mean, Mode and Median

Mean, mode and median are measures of central tendency. They are values around which data tends to cluster.

Mean

This is usually referred to as arithmetic mean, and is the average value for the data.

Example 2

Find the mean of the following marks:

9, 5, 5, 4, 5, 3, 5, 11, 6, 3, 6, 8, 9, 6, 13, 8, 8, 13, 5, 10

Solution

The mean score $= \dfrac{\text{total marks scored}}{\text{total number of students}}$

Sum $= 9+5+5+4+5+3+5+11+6+3+6+8+9+6+13+8+8+13+5+10$

Total number $= 142$

$\therefore$ mean score $= \dfrac{142}{20}$

$= 7.1$

We can use frequency distribution table to find the mean as follows:

Table 18.3

Marks (x)	*Frequency* (f)	fx
3	2	6
4	1	4
5	5	25
6	3	18
8	3	24
9	2	18
10	1	10
11	1	11
13	2	26
	$\Sigma f = 20$	$\Sigma fx = 142$

The symbol Σ is a Greek symbol (sigma). It means 'sum of'

Sum of the frequencies 'f' is written as Σf. Sum of products of x and f, is written as Σfx. The mean is denoted as $\overline{x}$.

Thus, $\overline{x} = \dfrac{\Sigma fx}{\Sigma f}$

From table 18.3, the mean mark, $\overline{x}$, is;

$$\begin{aligned}\overline{x} &= \frac{\Sigma fx}{\Sigma f} \\ &= \frac{142}{20} \\ &= 7.1\end{aligned}$$

Mode

This is the most frequent item or value in a distribution or data. For example in table 18.3, the mark 5 is the most frequent. It is called the mode. Its corresponding frequency is called the modal frequency. A data can have more than one mode. When a data has two most frequent items, it is said to be bimodal.

Example 3

The following are the times in seconds taken by athletes in a 100 m race:
10.0, 9.9, 9.9, 11.0, 10.0, 10.1, 12.0, 11.0, 10.3, 11.3, 11.0.
What is the mode?

Solution

The most frequent time is 11.0 seconds. The mode is 11. 0 seconds.

Median

The following were the marks obtained by 11 students in a mathematics test:
40, 35, 37, 50, 54, 39, 60, 57, 56, 51, 65.
To find the median, the marks are arranged in either ascending or descending order, that is; 35, 37, 39, 40, 50, **51**, 54, 56, 57, 60, 65, or
65, 60, 57, 56, 54, **51**, 50, 40, 39, 37, 35
The one at the middle is then the median.
Thus, 51 is the median.

If there is an even number of items, the average of the two in the middle is taken as the median. Consider the following ages in years of 6 form two students: 15, 14, 16, 14, 17 and 18.
To find the median, arrange the ages in either ascending or descending order, i.e., 14, 14, 15, 16, 17, 18

Then, the median age is $\frac{15 + 16}{2} = \frac{31}{2}$ years

$= 15\frac{1}{2}$ years

In general, to get median, arrange the items in order of size. If there are N items and N is an odd number, the item occupying $\left(\frac{N+1}{2}\right)^{th}$ position is the median. If

N is even, the average of the items occupying $\frac{N^{th}}{2}$ and $\left(\frac{N+1}{2}\right)^{th}$ positions is the median.

The measures of central tendency are useful in our everyday life activities. Below are some examples of situations in which these measures are applied:

(i) In our industries, e.g. shoe production firms, the modal size in use is manufactured in large quantities.

(ii) In the tourist industry, there is heavy preparation in the month when the highest number of tourists visit the country.

(iii) For a shopkeeper, the favourite items are bought in plenty.

(iv) For a hotelier, the favourite dish for customers will be important in preparing a menu.

(v) In a country where the mean age of a population is below 20 years, the government may find it necessary to plan for more schools, colleges and jobs.

(vi) The mean monthly rainfall of a particular region may determine the type of crops that can be grown.

Exercise 18.1

1. The data below represents marks scored by 10 students in a test. Find the mean, the mode and the median of the marks:
   7, 8, 9, 14, 13, 5, 10, 9, 11, 6
2. Forty eight pupils gave the number of children in their homes as follows:

3	4	1	2	3	2	2	1	3	5	6	8	6	1	4	2
3	1	5	7	8	9	1	2	1	2	5	6	5	5	7	8
2	3	1	3	2	11	1	2	4	2	3	3	4	2	1	3

   (a) Make a frequency distribution table for the data.
   (b) Determine the mode.
   (c) Calculate the mean number of children.
3. Twenty pupils in a class measured their masses and recorded them to the nearest kilogramme, as below:

40	48	56	52	49	57	56	52	53	48
38	39	43	47	41	60	63	59	45	51

   Make a distribution table and find the mean and the mode.
4. The classes in a primary school had the following number of pupils:

42	45	48	40	46	42	44	48	39	40	42
41	47	46	45	49	45	42	40	38	39	
46	47	42	40	41	43	44	45	46	48	

   (a) Make a frequency distribution table.

(b) Determine the mode.

(c) Calculate the mean.

5. In an experiment to measure the length, in centimetres, of an exercise book, 16 students recorded the following:

12.8	12.5	13.4	13.6	13.5	14.5	15.2	11.8
14.5	13.9	13.7	14.4	14.4	14.3	15.3	14.7

Construct a frequency table and use it to determine the mean and the mode.

6. A sample of 40 nuts were picked at random from a factory and their diameters measured to the nearest 0.1 mm as below:

20.4	20.5	10.1	10.2	20.3	20.0	19.9	19.9
20.6	20.5	20.4	20.3	20.1	20.9	20.2	20.3
20.3	20.4	20.1	20.1	20.8	20.7	20.5	20.4
20.0	20.1	20.5	20.6	20.7	20.1	20.0	19.8
20.3	20.4	19.9	20.0	21.0	20.9	20.8	20.4

(a) Prepare a frequency distribution table for the information.

(b) Find the mode, mean and median of the data.

7. In an athletics competition held in Kericho district, 12 athletes ran the 400 m race and their times in seconds were recorded as below:

58.2	54.2	49.8	52.3	55.6	50.8
53.8	51.9	50.7	52.3	57.2	52.3

Find the mean, mode and median.

8. The heights in metres of 50 children were recorded as follows:

1.21	1.25	1.36	1.42	1.53	1.67	1.78	1.60	1.50	1.40
1.34	1.42	1.52	1.44	1.53	1.42	1.72	1.66	1.62	1.32
1.30	1.24	1.46	1.38	1.39	1.42	1.50	1.62	1.38	1.44
1.33	1.44	1.28	1.35	1.40	1.70	1.80	1.36	1.72	1.50
1.40	1.36	1.38	1.39	1.50	1.61	1.70	1.28	1.60	1.80

make a frequency table and from it determine the mean, the mode and the median.

In numbers 9 to 11, find the mean, mode and median

9. The following frequency distribution table represents masses in kilogrammes of children in a class:

Table 18.4

Mass of children (kg)	35.5	36.8	38.2	40.0	40.2	41.0	41.4
No. of children	4	3	2	2	2	14	8

10. The volume of milk (to the nearest 0.01*l*) in 24 tetrapaks was measured and recorded as below:

0.50	0.50	0.49	0.48	0.54	0.50	0.47	0.51
0.49	0.51	0.53	0.52	0.52	0.50	0.48	0.52
0.48	0.48	0.50	0.50	0.50	0.49	0.51	0.53

11. The table below shows the duration, to the nearest minute, of telephone calls through an exchange:

Table 18.5

Duration in min.	0	1	2	3	4	5	6	7	8	9	10
Frequency	2	2	11	20	14	9	9	7	3	2	1

12. In an examination, the mean mark for Form 2B, in which there were 34 students, was 51. Form 2A, in which there were 36 students, had a mean of 42 marks. Calculate (to 1 dec. place) the mean mark for the two classes combined.
13. The mean of the numbers 3, 4, a, 5, 7, a, 5, 8, 5 and 9 is equal to the mode. Find the value of a and hence the median of the data.
14. The mean age in a Form II class of 36 boys was 16 years 8 months. If an 18-year-old boy later left the class, what was the new mean age for the class?
15. Forty five pupils in a mixed class sat for a mathematics test. The mean mark for the 25 girls in this class was 58.5 while the mean mark for all the pupils was 60.2. Calculate the mean mark for the boys.
16. The mean height of five girls was found to be 148 cm. When a sixth girl joined the group, the mean height was reduced by 3 cm. Find the height of the sixth girl.

18.4: Grouped Data

When a large number of items is being considered, it is necessary to group the data. Consider the masses in kilogrammes of 50 women in a clinic, which were recorded as follows:

63.7	50.0	58.0	73.4	76.3	42.3	58.0	45.0	62.1	78.0
58.0	59.5	45.2	79.6	69.1	42.8	49.8	84.2	54.9	52.6
52.1	63.5	58.0	45.5	58.7	59.5	40.6	61.4	43.5	59.7
72.9	49.2	74.2	60.2	43.9	73.4	53.3	67.3	77.7	63.5
62.3	46.8	66.2	52.1	59.7	60.5	60.3	69.6	80.2	54.4

The masses vary from 40.6 kg to 84.2 kg. The difference between the smallest and the biggest values in a set of data is called the **range**. In this case, the range is 84.2 – 40.6 = 43.6.

The data can be grouped into a convenient number of groups called **classes.** Usually, the convenient number of classes varies from 6 to 12. In this example, a class size of 4 will give us;

$\frac{43.6}{4} \simeq$ 11 classes.

A class size of 5 will give us $\frac{43.6}{5} \simeq$ 9 classes.

A class size of 6 will give us $\frac{43.6}{6} \simeq$ 8 classes.

Let us take a class size of 5 for the data.

Table 18.6

Class	40-44	45-49	50-54	55-59	60-64	65-69	70-74	75-79	80-84
Frequency	5	5	7	6	13	3	5	3	3

Note:

40-44 class includes all values equal to or greater than 39.5 but less than 44.5. Thus, 40 and 44 are called **class boundaries**, while 39.5 and 44.5 are called **class limits**. The class interval (width) is obtained by getting the difference between the class limits. In this case, $44.5 - 39.5 = 5$, is the class size or class interval.

The class with the highest frequency is called the **modal class**. In this case, it is 60-64 class, with a frequency of 13.

There are 50 women. To get the median, we need the masses of the 25th and 26th women when ranked from the lightest to the heaviest. Up to 59.5, there are 23 women. The 25th woman falls in the 60-64 class, in which there are 13 women.

Two additional women from this class will make up 25. We therefore take $\frac{2}{13}$ of the (60-64) class.

The estimate of the mass of the 25th woman;

$59.5 + (\frac{2}{13} \times 5) = 60.269$ kg

Similarly, the estimate of the mass of the 26th woman is;

$59.5 + (\frac{3}{13} \times 5) = 60.654$ kg

Therefore, the estimate of the median $= \left(\frac{60.269 + 60.654}{2}\right)$ kg

$= \left(\frac{120.923}{2}\right)$ kg

$=$ 60.5 kg (3 s.f.)

To calculate the mean from a grouped data, the midpoint of each class is taken to represent each item. For example, the midpoint of 40-44 is;

$$\left(\frac{40 + 44}{2}\right) = 42.0$$

The calculation procedure is set out as shown in the table below:

Table 18.7

Class	Midpoint (x)	Frequency (f)	xf
40-44	42.0	5	210
45-49	47.0	5	235
50-54	52.0	7	364
55-59	54.0	6	324
60-64	62.0	13	806
65-69	67.0	3	201
70-74	72.0	5	360
75-79	77.0	3	231
80-84	82.0	3	246
		$\Sigma f = 50$	$\Sigma fx = 2\,977$

Then, mean, $\bar{x} = \frac{\Sigma fx}{\Sigma f}$

$= \frac{2\,977}{50}$

$= 59.54$

$\therefore$ The mean mass is 59.54 kg

Grouping the data makes it less accurate. Therefore, the mean and the median obtained from grouped data are estimates.

Exercise 18.2

1. The examination marks in a mathematics test, for 72 students were as follows:

36	62	15	28	60	30	25	35	75
14	16	33	58	72	80	92	44	57
55	56	70	34	40	18	15	28	32
60	57	68	83	30	32	40	38	45
48	52	58	62	65	75	84	63	58
47	55	60	38	25	18	29	15	32
35	38	45	43	46	53	58	64	68
54	59	60	61	64	65	70	78	90

Using class interval of 10, that is 10 – 19, 20 –29, …, make a frequenc distribution table and from it:

(a) find the modal class.

(b) estimate:

(i) the mean.

(ii) the median.

2. The heights, to the nearest centimetre, of 50 students in a particular schoo were as follows:

165	170	182	169	165	180	182	175
180	184	174	186	175	175	186	180
174	183	172	167	168	158	188	190
135	148	159	148	182	163	140	145
156	158	155	147	143	142	138	150
160	156	140	148	158	165	180	175
184	168						

(a) Using class interval of 10, make a frequency distribution table :

(b) From the table:

(i) state the modal class.

(ii) estimate the median and the mean.

3. The populations in thousands of 42 districts were recorded as follows:

155	182	143	88	89	162	190
95	172	132	104	125	116	142
120	116	136	156	159	170	182
128	160	158	172	178	90	86
98	88	95	118	132	140	153
112	170	163	171	185	189	180

Make a grouped frequency distribution table and from it:

(a) find the modal class.

(b) estimate the mean and the median.

4. In a Kiswahili test, 40 students scored the following marks:

43	39	59	56	58	63	71	40
72	66	47	38	51	50	61	64
32	78	29	32	45	80	70	57
52	46	45	39	58	72	41	55
56	53	66	63	61	46	82	64

Group the above data using class intervals of size 5 and 25 – 29 as the first class. Hence, estimate the mean and the median. State also the modal class.

In questions 5 to 8, estimate the mean and the median, and state the modal class.

5. Masses of fish in kilogrammes caught by fishermen in one day:

Table 18.8

Mass in kg	0-4	5-9	10-14	15-19	20-24	25-29	30-39	40-44
No. of fish	2	6	20	12	10	5	6	2

6. The number of words which could be read per minute by 80 candidates in an examination:

Table 18.9

Words read	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Frequency	20	5	5	6	10	17	9	4	3	1

7. The diameters, to the nearest millimetre, of 78 bolts:

Table 18.10

Diameter (mm)	25-26	27-28	29-30	31-32	33-34	35-36	39-40	41-42	43-44
No. of bolts	2	4	7	18	20	15	8	3	1

8. The amounts, in shillings, of pocket money given to students of a particular school:

Table 18.11

Pocket money in Ksh.	100-190	200-290	300-390	400-490	500-590	600-690	700-790	800-890	900-990
No. of Students	5	14	40	38	24	20	15	10	4

9. The information below represents scores of 40 candidates in an aptitude test:

11.3	23.6	41.2	25.6	50.0	51.2	55.3	47.2
21.2	21.6	11.5	53.1	35.6	39.2	39.9	26.1
22.4	23.5	42.1	57.2	11.5	31.3	11.7	27.0
11.3	11.8	54.2	33.4	37.2	39.6	56.7	47.2
21.4	45.9	42.1	11.7	46.2	59.0	11.5	28.4

By using groups 10-14, 15-19, 20-24 ...:

(a) determine the modal class.

(b) estimate the mean and the median.

10. The data below represents times in seconds recorded in the heats of a 100 m race during an athletics meeting:

14.7	13.8	14.6	15.2	15.0	14.5	15.0	14.9	14.7	12.2
11.8	14.0	12.7	13.2	15.0	15.0	15.2	15.7	14.7	15.2
14.5	15.5	11.9	12.5	15.1	15.2	15.4	12.1	11.9	14.7
14.8	14.9	15.6	15.0	13.2	13.4	14.5	15.0	15.1	15.4
12.0	14.7	12.4	13.4	13.6	12.8	13.3	11.8	11.5	12.0

By using groups of 11.0 – 11.4, 11.5 – 11.9, 12.0 – 12.4 ….., make a frequency distribution table for the above data and hence:

(a) determine the modal class.

(b) estimate the mean and median.

18.5: Representation of Data

The main purpose of representation of statistical data is to make collected data more easily understood. Methods commonly used in representation are:

(i) Bar graphs.

(ii) Pictograms.

(iii) Pie charts.

(iv) Line graphs.

(v) Histograms.

(vi) Frequency polygons.

Bar Graphs

A bar graph consists a number of spaced rectangles which generally have major axes vertical. Bars are of uniform width. The axes must always be labelled and scales indicated. For example, the Kamau's performance in five subjects was as follows:

Maths	80
English	75
Kiswahili	60
Biology	70
Geography	75

This information can be represented on a bar graph as shown below:

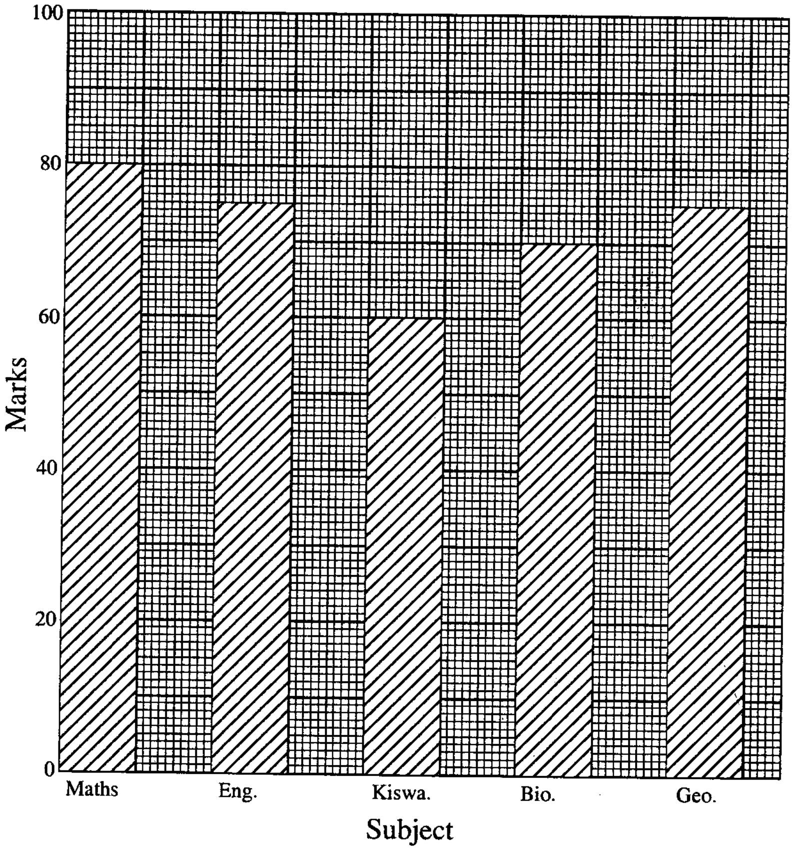


Fig. 18.1

Pictograms

In a pictogram, data is represented using pictures. Each picture represents a given quantity. Consider the following table which shows the different makes of cars imported over a certain period:

Toyota	2 000
Nissan	3 500
Peugeot	1 500

This information can be represented in a pictogram using the following key;

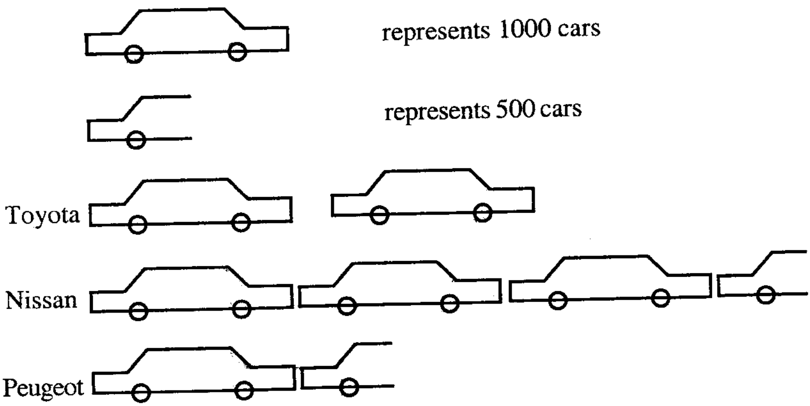


Fig 18.2

Pie chart

A pie chart is a circle divided into various sectors. Each sector represents a certain quantity of the item being considered. The size of the sector is proportional to the quantity it represents.

Consider the total population of animals in a farm given as 1 800. Out of these 1 200 are chicken, 200 cows, 300 goats and 100 ducks. This information can be represented in a pie chart as follows;

Chicken

$$\text{Angle} = \frac{\text{Number of chickens}}{\text{total population}} \times 360$$

$$= \frac{1\,200}{1\,800} \times 360$$

$$= 240^0$$

Cows

$$\text{Angle} = \frac{200}{1\,800} \times 360$$

$$= 40^0$$

Goats

$$\text{Angle} = \frac{300}{1\,800} \times 360$$

$$= 60^0$$

Ducks

$$\text{Angle} = \frac{100}{1\,800} \times 360$$

$$= 20^\circ$$

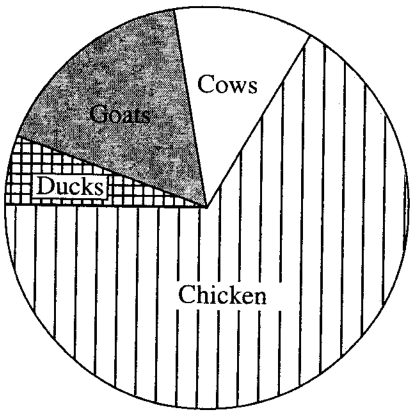


Fig. 18.3

Line Graphs

In line graphs, data is represented using lines. Examples of line graphs are a graph showing temperature of a patient at various times of the day, and a travel graph.

Consider the following table which shows the temperature of a patient at different time of the day:

Table 18.12

Time	8 a.m.	9 a.m.	10 a.m.	11 a.m.	12 noon	1 p.m.	2 p.m.
Temp. (°C)	35.6	36.4	37.0	37.2	36.8	35.9	37.1

The data can be represented in a line graph as below:

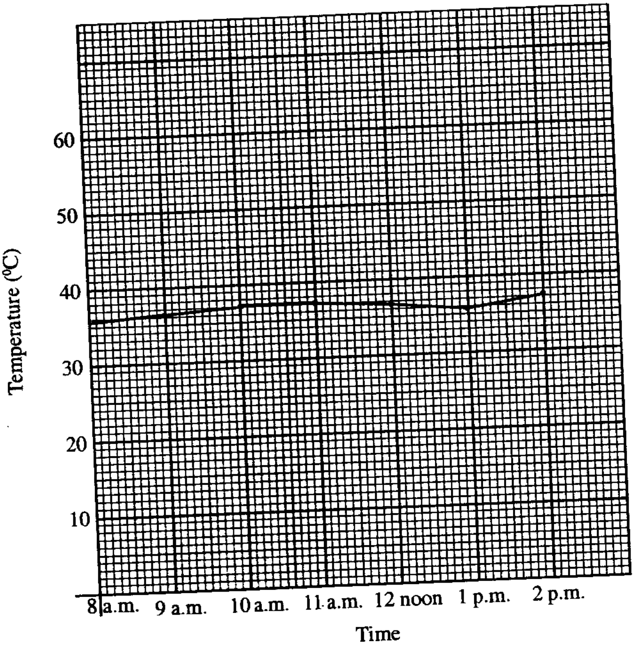


Fig. 18.4

Histograms

In this, the frequency in each class is represented by a rectangular bar whose area is proportional to the frequency. When the bars are of same width the height of the rectangle is proportional to the frequency. Note that the bars are joined together.

For example, the following data represents masses, to the nearest kilogramme, of fish caught by a fisherman in a day:

Table 18.13

Mass (kg)	5-9	10-14	15-19	20-24	25-29	30-34	35-39	40-44
No. of fish	6	20	12	10	5	6	2	1

The above data can be represented in a histogram as shown in figure 18.5.

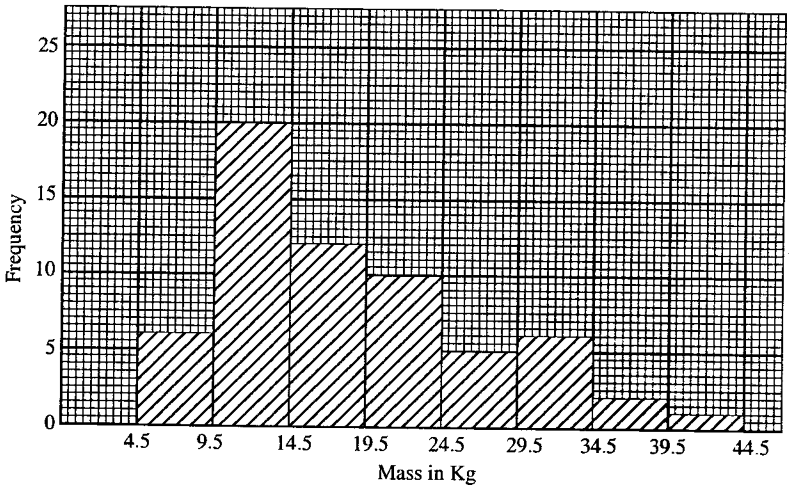


Fig. 18.5

Note:

The class boundaries mark the boundaries of the rectangular bars in the histogram

Histograms can also be drawn when the class intervals are not the same. For example, the following table shows the marks obtained in a maths test:

Table 18.14

Marks	10-14	15-24	25-29	30-44
No. of students	5	16	4	15

The above information can be represented in a histogram as shown below:

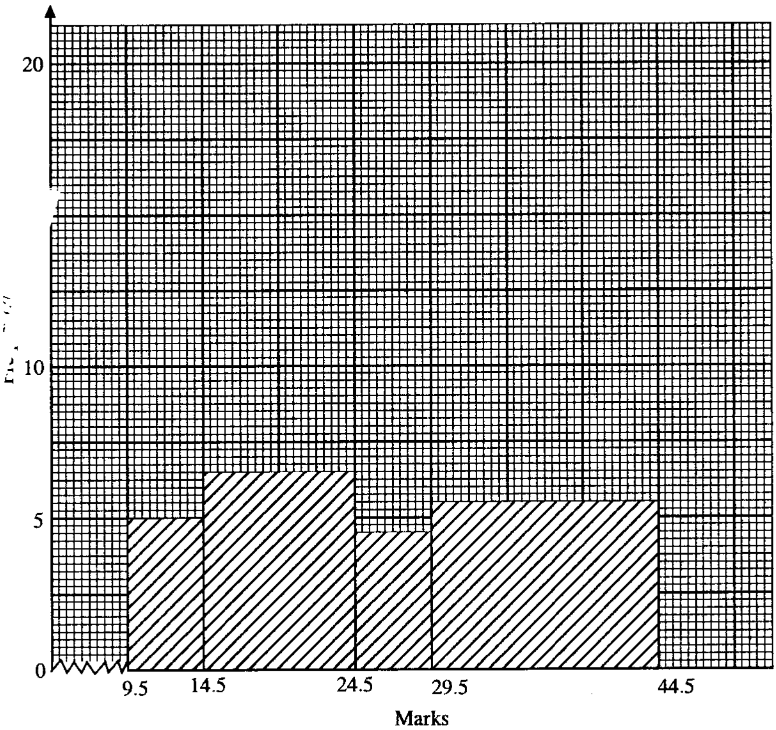


Fig. 18.6

Note:
When the class is doubled, the frequency is halved.

Frequency Polygon
When the midpoints of the tops of the bars in a histogram are joined with straight lines, the resulting graph is called a frequency polygon. However, it might not be necessary to draw bars to get a frequency polygon. You can obtain a frequency polygon by plotting the frequency against the midpoints of the classes.

Figure 18.7 represents a frequency polygon for the data below in table 18.13.

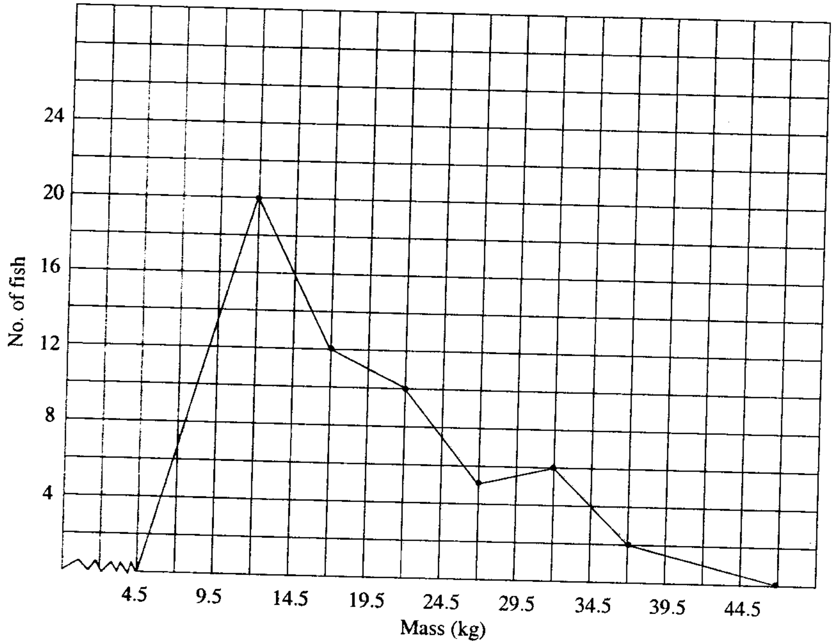


Fig. 18.7

More than one frequency polygon can be drawn on the same axes for the purpose of comparison. For example, in an examination, the marks in Mathematics and Physics were grouped as given below. We can draw frequency polygons to represent the data as in figure 18.8.

Table 18.15

	Marks	10-19	20-29	30-39	40-49	50-59	60-69	70-79
Frequency	Mathematics	2	6	7	13	6	4	2
	Physics	3	9	10	6	9	2	1

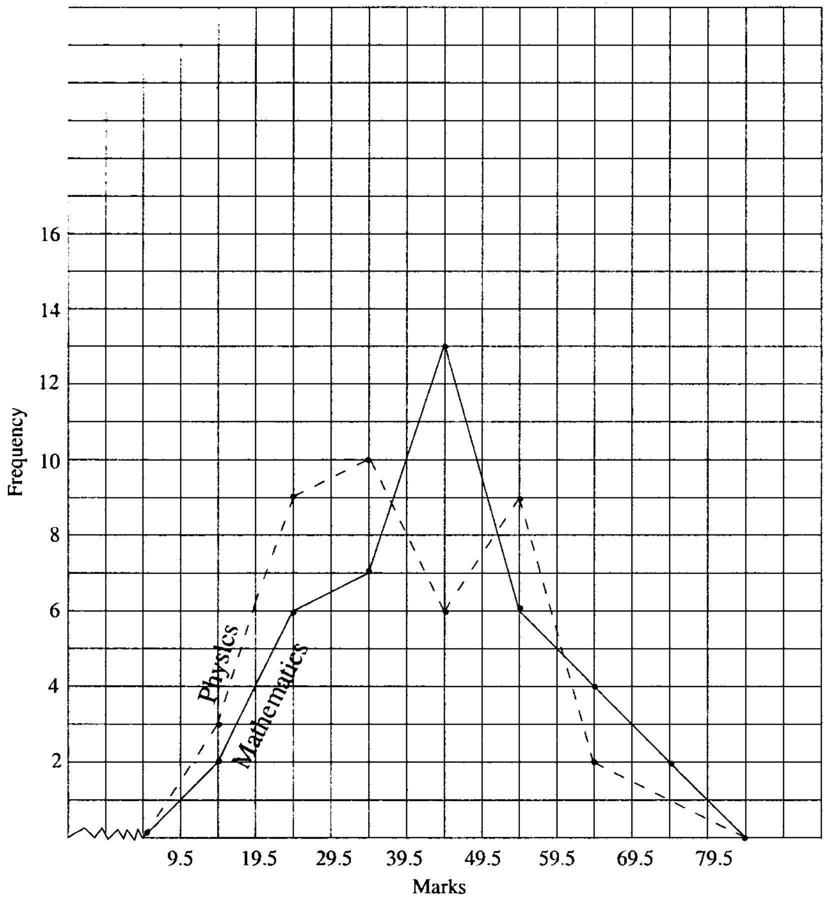


Fig. 18.8

From the above frequency polygon, we notice that:

(i) there were more students in Physics than in Mathematics with less than 39.5 marks.

(ii) there were more students in Mathematics than in Physics with more than 39.5 marks. This implies that more students had relatively higher marks in Mathematics than in Physics.

From the two observations, we can conclude that the performance in Mathematics was better than that in Physics.

Exercise 18.3

1. A 24-hectare farm is divided as follows:

   Coffee – 4 ha
   Grass – 3 ha
   Maize – 7 ha
   Bananas – 5 ha
   Homestead – 0.5 ha
   Vegetables and paths – 4.5 ha

   Draw a pie chart to represent this information.

2. A test given to 40 students produced the following results:

   5 students got grade A.
   12 students got grade B.
   15 students got grade C.
   5 students grade D.
   3 students got grade E.

   (a) Show the above information on a pie chart.
   (b) Find the percentage which represents:
      (i) those with grade A.
      (ii) those with grade C and D.

3. The following table shows the number of trees planted in a certain farm:

   Table 18.16

Year	1998	1999	2000	2001	2002
No. of trees	7 400	11 200	10 700	5 600	9 800

   Represent the information on a bar graph.

4. The data below shows the population of Matunya Secondary School.

   Table 18.17

Year	1996	1997	1998	1999	2000
No. of Students	200	300	275	400	350

   Draw a pictogram to represent this information.

5. The table below shows the importation of vehicles (in thousands) for the year 1994 to 2002:

   Table 18.18

Year	1994	1995	1996	1997	1998	1999	2000	2001	2002
No. of vehicles (in thousands)	15	24	29	42	50	48	45	43	38

Draw a line graph to represent the information.

6. The heights, to the nearest centimetre, of 20 boys and 20 girls were measured and recorded as follows:

Boys:	119	122	123	128
	129	130	131	132
	132	135	136	137
	138	140	142	145
	150	152	154	160
Girls:	116	120	121	125
	126	127	129	130
	130	132	135	135
	136	138	141	144
	148	150	152	159

(a) Construct a histogram for each of the above data.
(b) Construct frequency polygons of the data on the same axes.
(c) What can you say about the heights of the boys and girls?

7. The table below shows the number of letters collected from the post office by a school driver during school year:

Table 18.19

Letters per day	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55
Frequency	5	19	21	23	25	27	20	25	13	12

Draw a frequency polygon to represent the data.

8. The number of pupils present in a class for 20 school days were:

43	38	39	37	33
31	28	35	27	32
29	30	34	44	24
32	34	41	36	38

(a) Make a grouped frequency table with a suitable class width.
(b) Draw a bar-graph to represent the information.
(c) Draw a histogram to represent the information.

9. Pupils in a class in a certain day school were asked how they travelled school. They gave the following information:

Table 18.20

Means of travel	Walk	Cycle	Matatu	Bus
No. of pupils	17	5	18	2

Draw a pie-chart to represent this data.

10. In a day school, the distances in kilometres travelled to school by some pupils were recorded as below:

Table 18.21

Distance in km	0-2	3-5	6-8	9-11	12-14	15-17	18-23
No. of pupils	20	6	9	8	7	4	8

Draw a histogram to represent this data.

11. The data below shows the total number of KCSE subjects passed in two schools:

Table 18.22

No. of subjects		0	1	2	3	4	5	6	7	8
No. of Candidates	School A	1	1	1	10	12	15	12	7	3
	School B	3	2	3	5	6	9	13	12	7

(a) On the same axes, draw frequency polygons for the two schools.

(b) If anyone with 6 or more passes qualified for higher education, how many candidates qualified in each school?

(c) If anyone with 4 or more passes could be admitted to a training college, how many qualified for a place in training colleges from each school?

ANGLE PROPERTIES OF A CIRCLE

19.1: Arc, Chord and Segment of a Circle

Arc

Figure 19.1 shows circle KLMN with centre O. Any part of the circumference of a circle is called an **arc**. Arc LMN is less than half the circumference of the circle and is therefore called the **minor arc**. The remaining part, LKN, which is greater than half the circumference of the circle, is called the **major arc.**

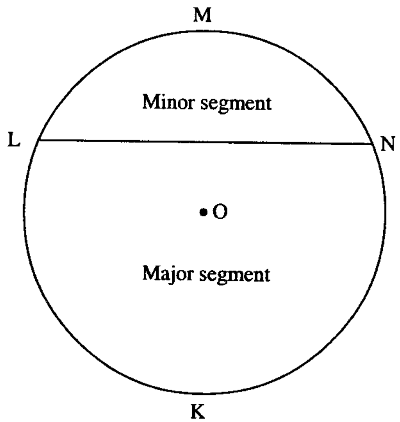


Fig. 19.1

Chord

A line joining any two points on the circumference is called a **chord**. For example, LN in figure 19.1 is a chord. The diameter of a circle is its longest chord.

Segment

Any chord divides the circle into two regions called **segments**. The smaller one is called the **minor segment** and the larger the **major segment**, see figure 19.1.

19.2: Angle at the Centre and Angle on the Circumference

Any angle which lies either at the centre or on the circumference of a given circle stands astride a chord or on an arc. The angle is said to be **subtended** by the chord or the arc. For example, in figure 19.2, arc NML subtends an angle of 120^0 at the centre of the circle.

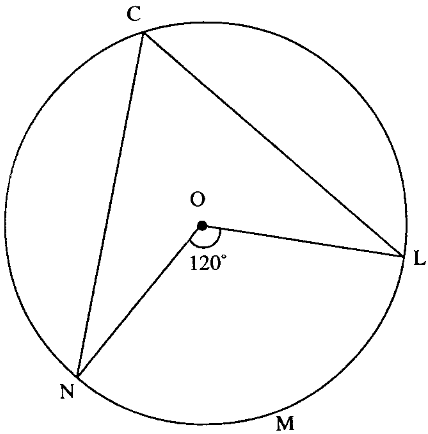


Fig. 19.2

Similarly, ∠NCL is subtended on the circumference by the same arc NML.

Draw the figure accurately and measure ∠NCL. What do you notice? Draw a circle of any reasonable radius with an arc subtending an angle at the centre and several angles at the circumference. Compare the sizes of angles at the circumference and at the centre. What do you notice?

You should have noticed that **the angle which an arc subtends at the centre is twice that it subtends at any point on the circumference of the circle.**

In figure 19.3, AB is the diameter of the circle with centre O.

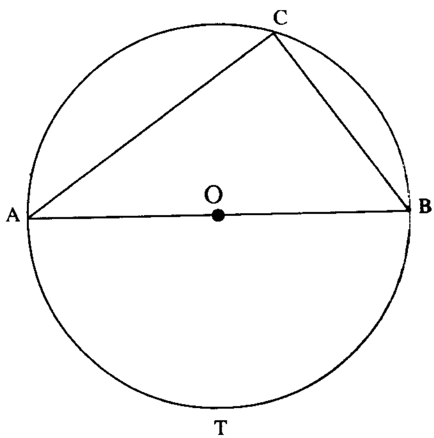


Fig. 19.3

The arc ATB subtends an angle of 180^0 at the centre of the circle. The same arc subtends $\angle ACB$ on the circumference.
But angle subtended at the centre $= 2 \times$ angle subtended on the circumference
Therefore:

$$\angle AOB = 2 \times \angle ACB$$
$$= 180^0$$
$$\angle ACB = \frac{180^0}{2}$$
$$= 90^0$$

Hence, the diameter of a circle subtends a right angle at any point on the circumference of the circle. Conversely, if a chord subtends a right angle at any point on the circumference of the circle, then the chord must be a diameter of the circle.

Figure 19.4 is a circle LMNPQRS with centre O.

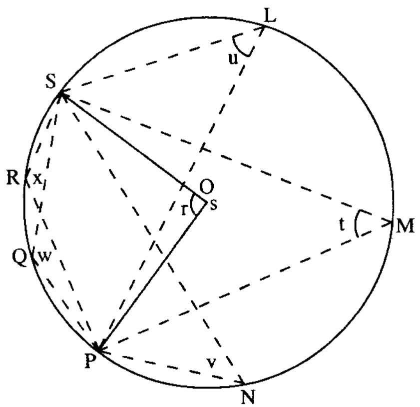


Fig. 19.4

(i) Name the arcs that subtend the angles marked t, u, v, w and x on the circumference.
(ii) Name the arcs that subtend the angles marked r and s at the centre O.
(iii) If $r = 120^0$, find u, t, v, w and x.

Example 1
In figure 19.5, show that $\angle BOC = 2\ \angle BAC$, if O is the centre of the circle.

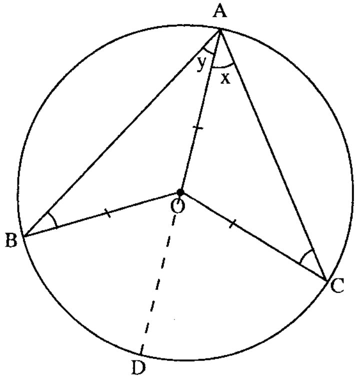


Fig. 19.5

Solution

In the figure, ∠ABO = ∠BAO = y (base angles of isosceles ΔABO).
∠ACO = ∠CAO = x (base angles of isosceles ΔACO)
∠BOD = ∠ABO + ∠BAO = 2y (exterior angle of a triangle)
∠COD = ∠ACO + ∠CAO = 2x (exterior angle of a triangle)
∠BOC = 2y + 2x
= 2(x + y)
But ∠BAC = x + y
Therefore, ∠BOC = 2∠BAC

Example 2

Find the values of x, y and z in figure 19.6 if O is the centre of the circle and ∠ABC = 30^0:

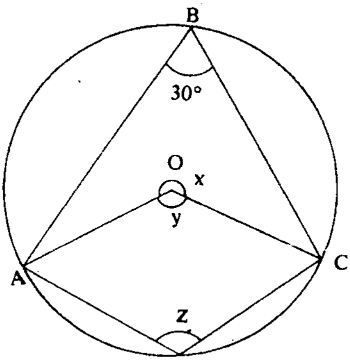


Fig 19.6

Solution

$y = 30° \times 2$

$= 60°$ (angle at the centre is twice the angle on the circumference)

$x = 360° - 60°$

$= 300°$

$z = \frac{300°}{2}$

$= 150°$

Exercise 19.1

1. Copy and complete the table below. All angles are subtended by the same arc.

Table 19.1

Angle at the centre	80°	64°		122°			110°		180°	270°	
Angle on the circumference		32°	42.5°		22.6°	62°	55°	110°			x

2. In figure 19.7, O is the centre of the circle. Find $\angle AOB$.

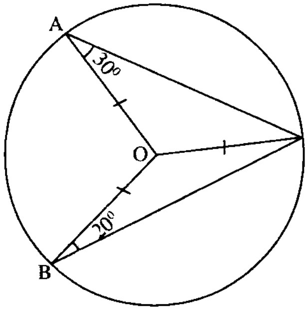


Fig. 19.7

3. Figure 19.8 shows a circle with the centre at O:

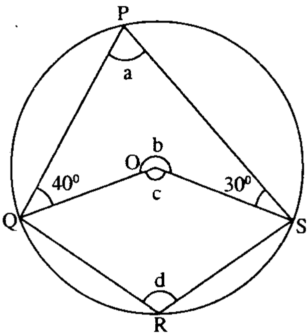


Fig. 19.8

Find the values of a, b, c and d if $\angle PQO = 40^0$ and $\angle PSO = 30^0$.

4. In figure 19.9, O is the centre of the circle. Show that the reflex angle $\angle POR = 2\angle PQR$:

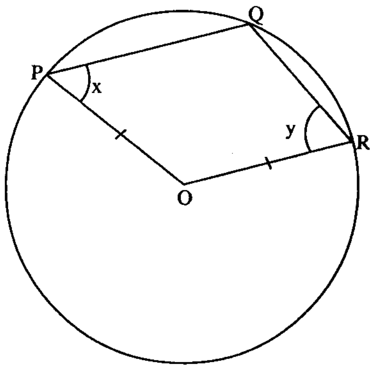


Fig. 19.9

5. In figure 19.10, O is the centre of the circle. LN and PM are diameters. Show that LM is parallel to PN:

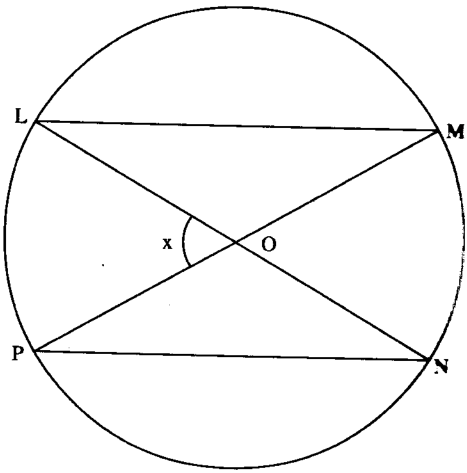


Fig. 19.10

6. In figure 19.11, O is the centre of the circle. Calculate x.

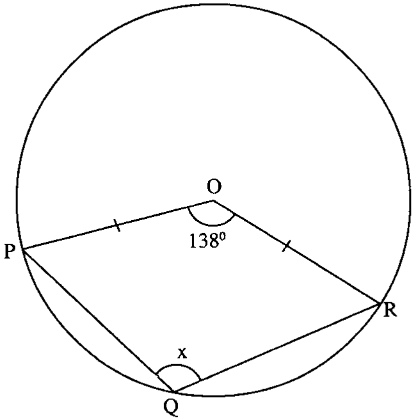


Fig. 19.11

7. In figure 19.12, O is the centre of the circle and DOB is the diameter. Find x and y if ABCO is a rhombus and $\angle AOD = 120^0$:

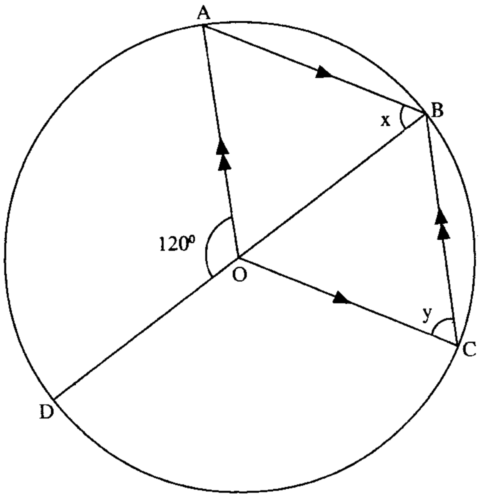


Fig. 19.12

8. In figure 19.13, O is the centre of the circle. BOD is the diameter, AC = BC and $\angle BAC = 25^0$. Find the size of:
   (a) $\angle AOD$.
   (b) $\angle BAO$.

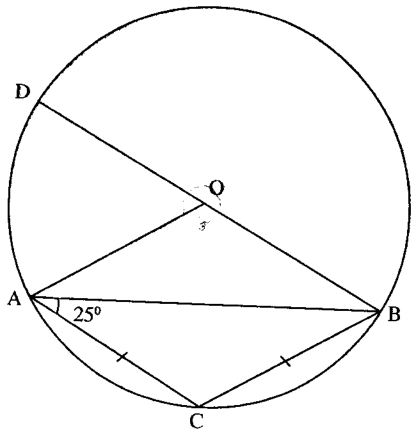


Fig. 19.13

9. AB and CD are intersecting chords of a circle with centre O. If AB is the diameter of the circle, show that:
   (a) $\angle AOC = 2\angle ABC$.
   (b) $ADB = 90^0$.
10. Figure 19.14 shows a circle, centre O, with two chords AB and CD intersecting outside the circle at X. If AD is the diameter, show that:
   (a) $\angle BDC = (90 + a)^0$.
   (b) $\angle CAX = \angle BDX$.

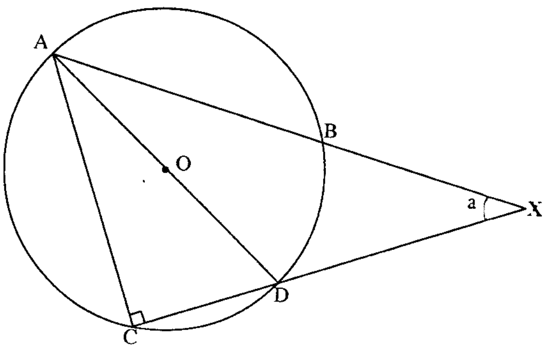


Fig. 19.14

11. Two circles with centres O and R intersect at P and Q as shown in figure 19.15. OR cuts the circles at S and T respectively. If $\angle AOR = 150^0$ and $\angle BRO = 120^0$, find a, b and c.

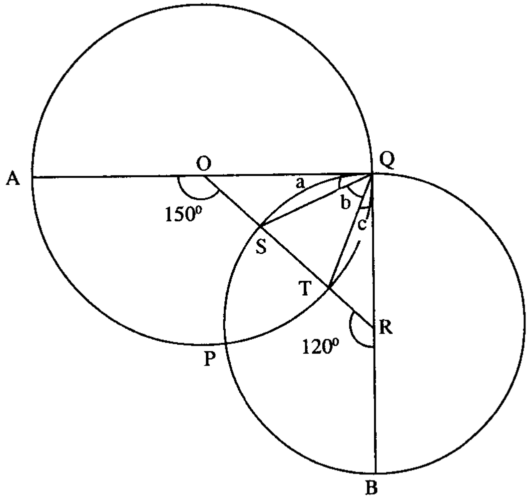


Fig. 19.15

12. In a circle KLMN with centre O, the diagonals of quadrilateral KLMN intersect at R. $\angle LKM = 38^0$, $\angle LNK = 64^0$ and $\angle KRN = 82^0$. Find:
    (a) $\angle KNM$.
    (b) $\angle LMN$.
    (c) $\angle KLM$.
13. Figure 19.16 shows a circle ABCD, centre O. BO is perpendicular to the diameter AOD and $\angle COD = 30^0$. Calculate:
    (a) $\angle BCA$.
    (b) $\angle CAD$.

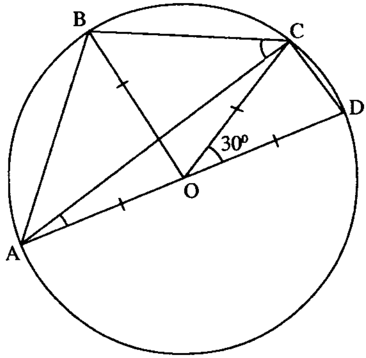


Fig. 19.16

14. ACB is a circle, centre O, with diameter AB perpendicular to BD as shown in figure 19.17. If BC = BD = CD and DO is perpendicular to BC:
    (a) show that triangles ABC and BOD are congruent.
    (b) find p and t.

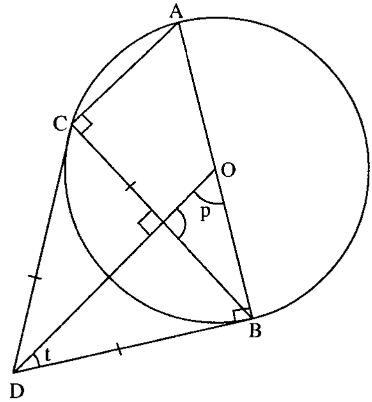


Fig. 19.17

19.3: Angles in the Same Segment

In figure 19.18, chord PQ divides the circle two segments:

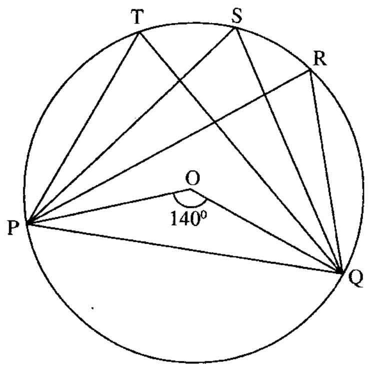


Fig. 19.18

The arc PQ subtends $\angle POQ$ at the centre and angles PTQ. PSQ and PRQ on the circumference in the major segment.

If $\angle POQ$ is 140^0, then $\angle PSQ = \angle PRQ = \angle PTQ = 70^0$ (angle subtended at the centre is twice angle subtended on the circumference). In general, **angles subtended on the circumference by the same arc in the same segment are equal.** Also note that **equal arcs subtend equal angles on the circumference**

Example 3

Calculate the value of a, b, c, d and e in figure 19.19:

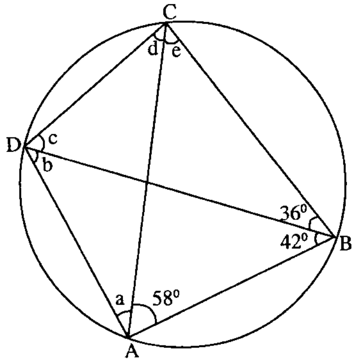


Fig. 19.19

Solution

$a = 36^0$, since $\angle DAC$ and $\angle DBC$ are subtended by the same arc DC.

$c = 58^0$, since $\angle CAB$ and $\angle CDB$ are subtended by the same arc BC.

$d = 42^0$, since $\angle DBA$ and $\angle DCA$ are subtended by the same arc DA.

$e + 58^0 + 42^0 + 36^0 = 180^0$ (angle sum of ΔABC.)

$e + 136^0 = 180^0$

$e = 44^0$

But e = b, since $\angle ADB$ and $\angle ACB$ are subtended by the same arc AB or are in the same segment that is major segment ADCB.

Therefore, $b = 44^0$.

Exercise 19.2

1. Find the values of a and b in figure 19.20:

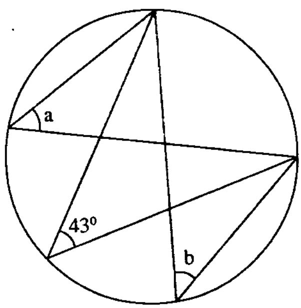


Fig 19.20

2. In figure 19.21, calculate the angles marked v, w, x, y and z:

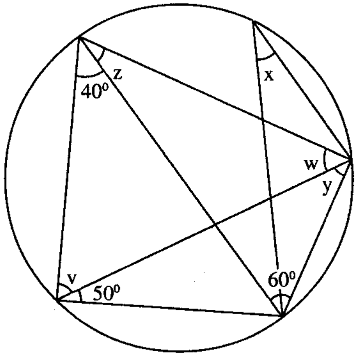


Fig. 19.21

3. In figure 19.22, O is the centre of circle ABCDE and BOE is the diameter: $\angle COD = 80^0$ and CB = CD.

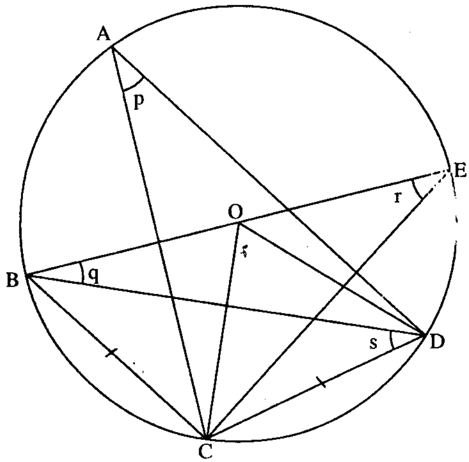


Fig. 19.22

Calculate the angles marked p, q, r and s.

4. In figure 19.23, O is the centre of the circle. Find the values of x and y.

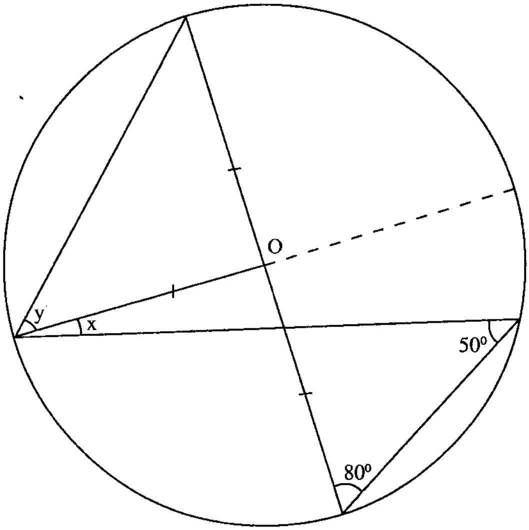


Fig. 19.23

5. In figure 19.24, O is the centre of the circle and POM is the diameter. If $\angle QLK = 73^0$, calculate $\angle QNM$:

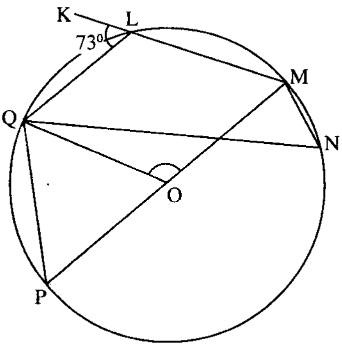


Fig. 19.24

6. Find the angles marked by small letters in the figure 19.25:

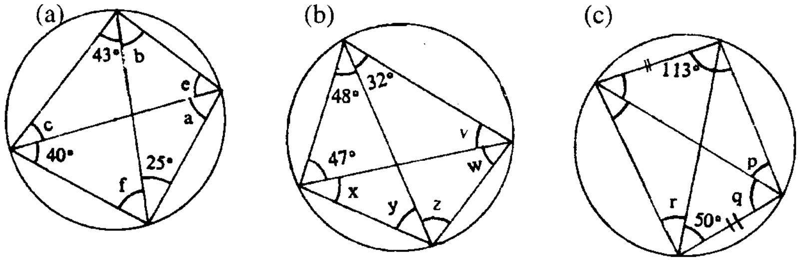


Fig. 19.25

7. In figure 19.26, O is the centre of the circle. Calculate the value of angle a:

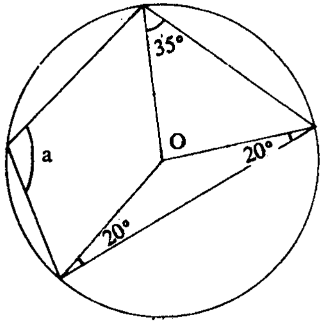


Fig. 19.26

8. Find the value of x and y in figure 19.27:

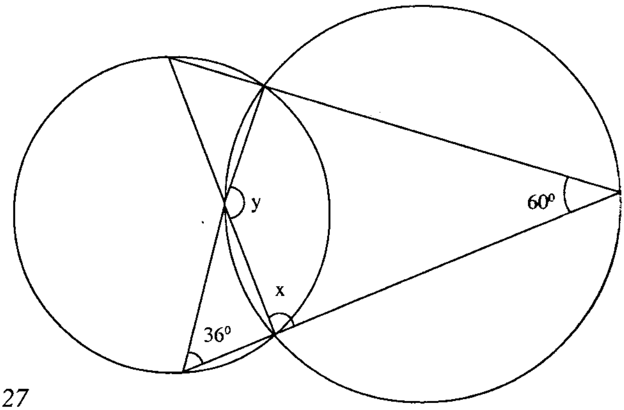


Fig. 19.27

9. Two equal chords WX and YZ are produced to meet at T. If $\angle WYX = 73^0$ and $\angle YZW = 37^0$, show that $\angle TZX = \angle XWY = 70^0$.

19.4: Cyclic Quadrilaterals

Figure 19.28 shows a quadrilateral ABCD with all its vertices lying on the circumference. Such a quadrilateral is said to be **cyclic** and the four vertices A, B, C and D are called **concyclic points.**

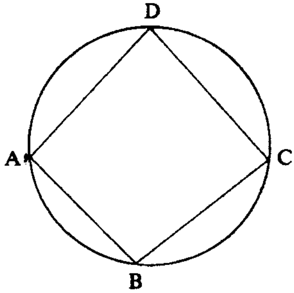


Fig. 19.28

Angle properties of a cyclic quadrilateral

Supplementary Angles

In figure 19.29, KLMN is a cyclic quadrilateral and O is the centre of the circle.

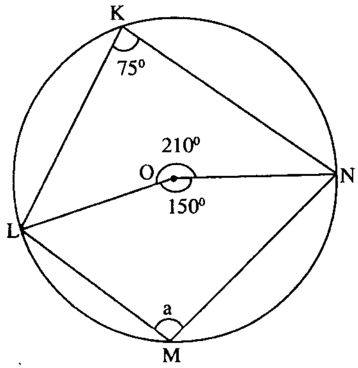


Fig. 19.29

The obtuse angle LON = 150^0.

$\angle$NKL = 75^0 (angle at the centre is twice the angle at the circumference).

The reflex angle LON = 210^0 (angles at a point add up to 360^0). What is the size of the angle marked a? What is the sum of $\angle$LKN and $\angle$LMN? What can you deduce about the sum of $\angle$KLM and $\angle$MNK?

You will notice that $\angle$LKN and $\angle$LMN add up to 180^0. Hence, angles KLM and MNK add up to 180^0.

The above deductions are true for any cyclic quadrilateral. Hence, the opposite angles of a cyclic quadrilateral are supplementary. Conversely, if the opposite angles of a quadrilateral inscribed in a circle add up to 180^0, then the quadrilateral is cyclic.

Exterior Angles

Figure 19.30 shows a cyclic quadrilateral PQRS with QR produced to T.

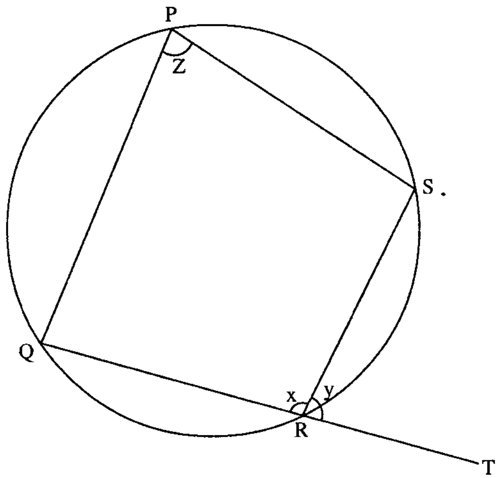


Fig. 19.30

Let $\angle QRS = x$, $\angle SRT = y$ and $\angle QPS = z$.

$x + y = 180^0$ (why?)

$x + z = (180^0$ why?)

$y = 180^0 - x$

$z = 180^0 - x$

Therefore, $y = z$

That is, $\angle QPS = \angle SRT$

This shows that if **a side of a cyclic quadrilateral is produced, the exterior angle formed is equal to the interior opposite angle.**

Example 4

Calculate the value of a and b in figure 19.31:

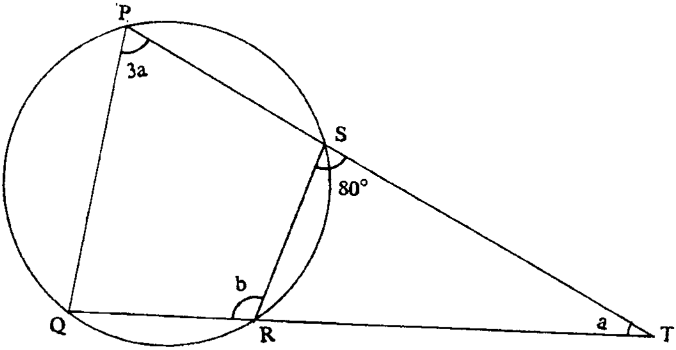


Fig. 19.31

Solution

$\angle PQR = \angle RST = 80^0$ (exterior angle of a cyclic quad.)

Therefore, $3a + a + 80^0 = 180^0$ (angle sum of ΔPQT)

$4a = 100^0$

$a = 25^0$

$3a + b = 180^0$ (opposite angles of a cyclic quad.)

Therefore;

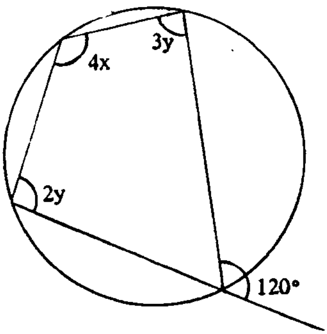
$75^0 + b = 180^0$

$b = 105^0$

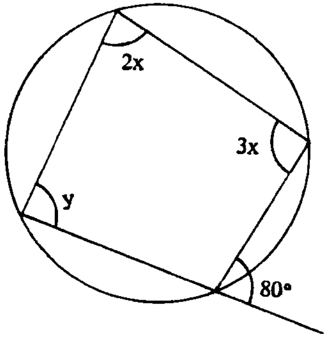
Exercise 19.3

1. Find the values of x and y in figure 19.32:

(a)



(b)



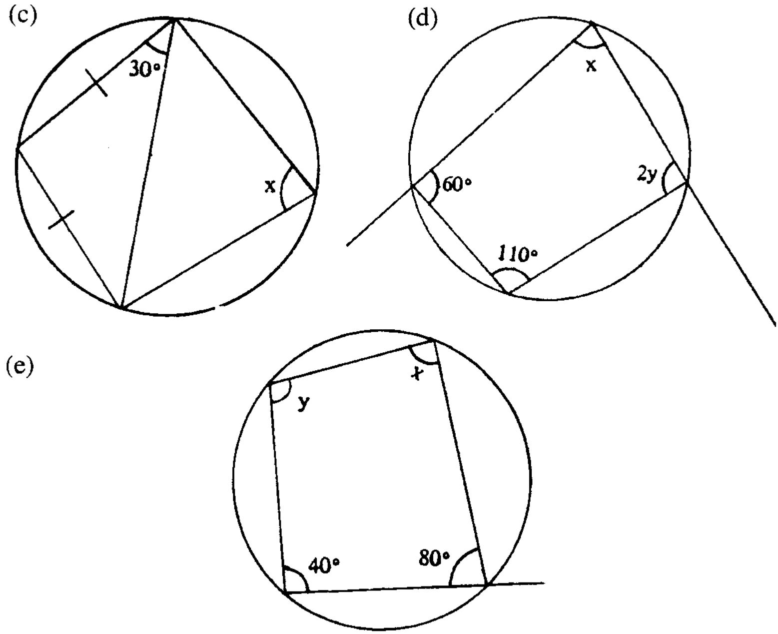


Fig. 19.32

2. In figure 19.33, calculate $\angle LNP$ if $\angle PLM = 70^0$ and $\angle KNL = 42^0$:

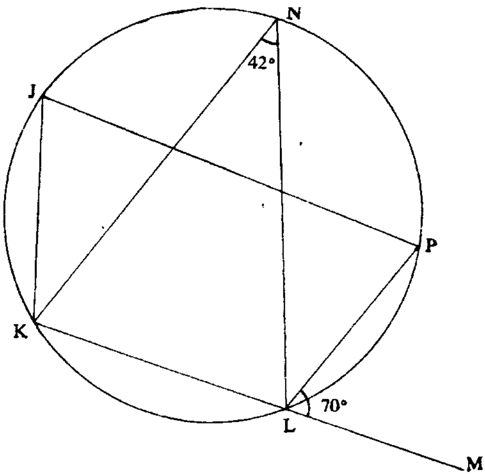


Fig. 19.33

3. In figure 19.34, O is the centre of the circle. The reflex angle AOD = 256^0 and $\angle ACE = 50^0$. Calculate:
   (a) $\angle BDC$.
   (b) $\angle BAE$.

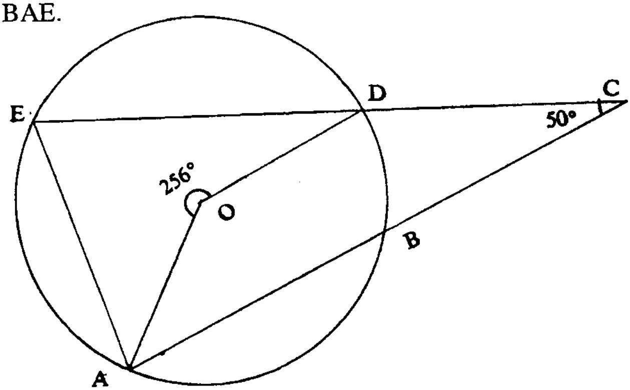


Fig. 19.34

4. In figure 19.35, $\angle VUZ = 110^0$ and $\angle XVY = 30^0$. Find the value of $\angle VWY$.

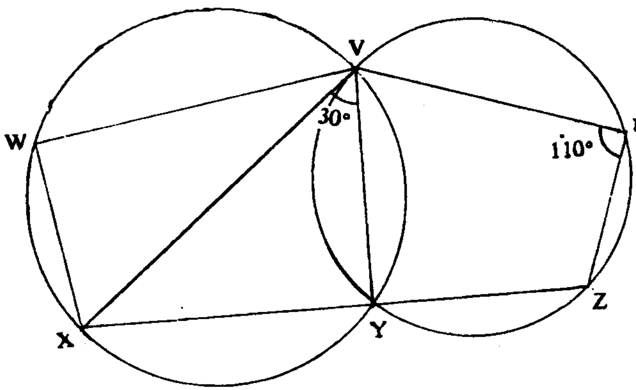


Fig. 19.35

5. PQRS is a cyclic quadrilateral. $\angle QPR = 32^0$, PS = SR, PQ = PR and T is a point on the circumference between R and S. Calculate $\angle RTS$.
6. In figure 19.36, O is the centre of the circle and BA is parallel to CD. If $\angle BAC = 31^0$, calculate $\angle BCD$ and $\angle ADE$.

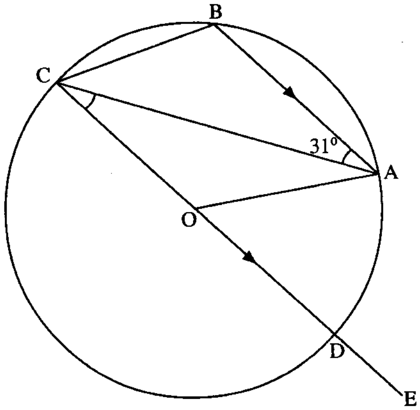


Fig. 19.36

7. In figure 19.37, WXYZ is a cyclic quadrilateral and VWZ a straight line. Angle VWX = 97^0 and $\angle XZY = 43^0$. Calculate the size of angle ZXY:

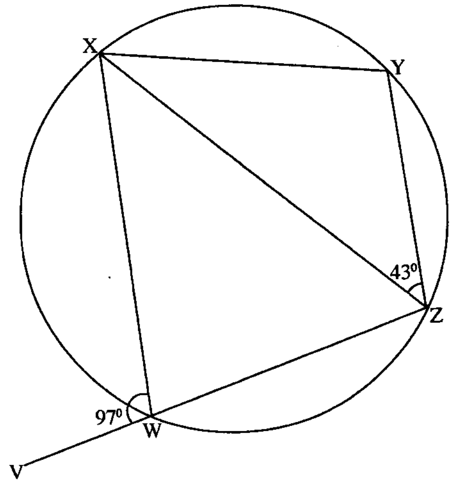


Fig. 19.37

Chapter Twenty

VECTORS

20.1: Introduction

Figure 5.1 shows two towns, A and B:

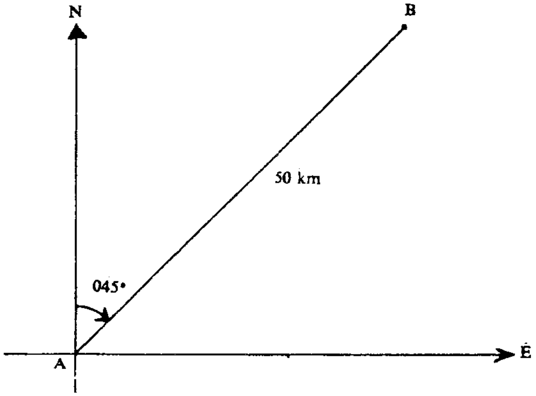


Fig. 20.1

Which of the following statements best describes the position of B relative to A?

(i) B is 50 kilometres from A.

(ii) B is north-east of A.

(iii) B is 50 kilometres north-east of A.

Clearly, the position of B relative to A is best described by (iii), which states not only the distance, but also the direction. A quantity which has both **magnitude** and **direction** is a **vector.** In the example, we have used both distance (magnitude) and direction to specify the position of B relative to A. Some examples of vector quantities are velocity, acceleration and force.

A quantity which has magnitude only is called a **scalar**. Some examples of scalar quantities are mass, distance, temperature and time.

20.2: Representation of Vectors

A vector can be represented by a directed line, as shown below:

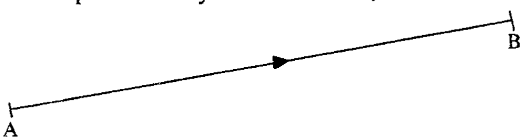


Fig. 20.2

The **direction** of the vector is shown by the arrow. Its **magnitude** is represented by the length of AB. Vector AB can be written in short as $\underset{\sim}{AB}$ or $\overrightarrow{AB}$ or **AB**. Its magnitude is denoted by |**AB**|. A is called the **initial point** and B the **terminal** point. Sometimes a vector can be denoted by a single small letter **a** or $\underset{\sim}{a}$ as shown in figure 20.3:

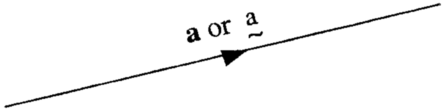


Fig. 20.3

In print, we use bold face **a**, but in writing, we use the notation $\underset{\sim}{a}$ to represent the 'vector a.'

20.3: Equivalent Vectors

Two or more vectors are said to be equivalent if they have:

(i) equal magnitude.

(ii) the same direction.

(a)

(b)

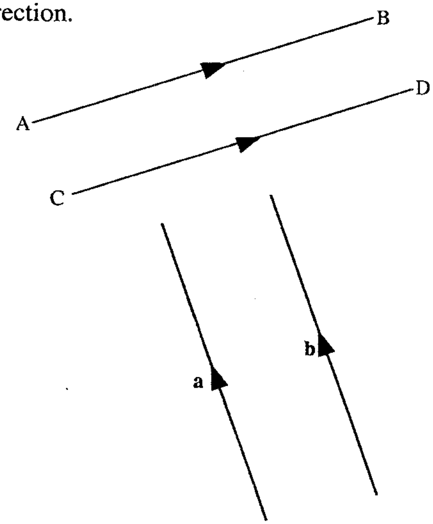


Fig. 20.4

In figure 20.4(a), **AB** = **CD** because:

(i) |**AB**| = |**CD**|.

(ii) they have the same direction.

Similarly, in figure 20.4(b), **a** = **b** because:

(i) $|\mathbf{a}| = |\mathbf{b}|$.

(ii) they have the same direction.

Figure 20.5 is a cuboid. **EF** = **HG.** Name the vectors equivalent to **CB, CD** and **CH**:

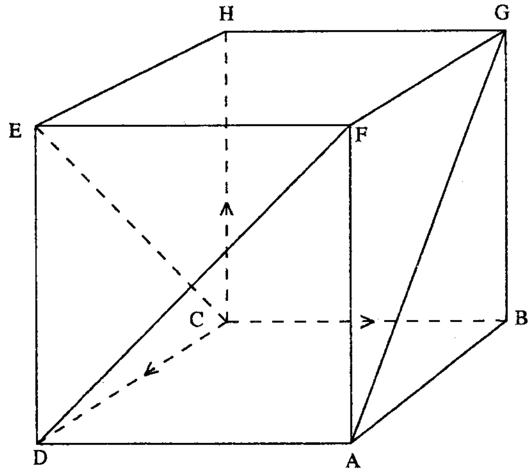


Fig. 20.5

20.4: Addition of Vectors

A movement on a straight line from point A to B can be represented using a vector. This movement can also be referred to as **displacement.**

Consider the displacement from A to B followed by that from B to C, see figure 20.6.

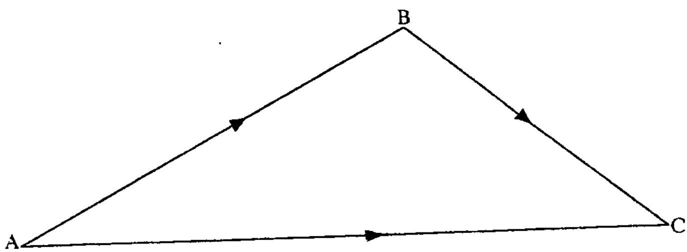


Fig. 20.6

The resulting displacement is **AC**. This is usually written as **AB** + **BC** = **AC.**

In order to add any two vectors, a re-arrangement may be necessary. The initial point of either vector is placed in the terminal point of the other, as illustrated in the following figures.

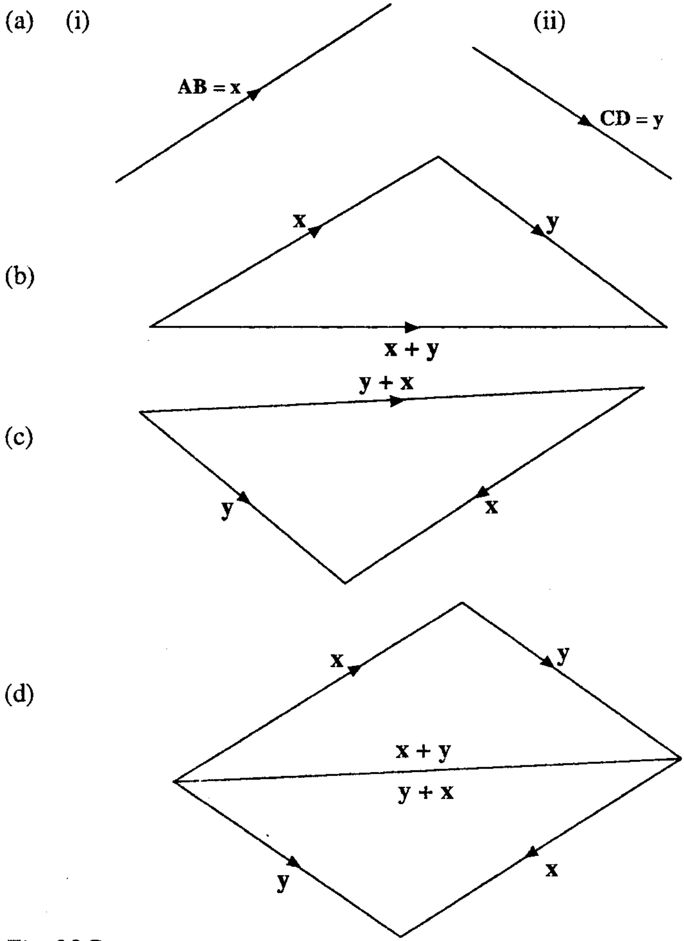


Fig. 20.7

The two ways of adding vectors AB and CD in figure 20.7(a) are shown in figure 20.7(b) and (c). In figure 20.7 (b), the initial point of **y** is placed on the terminal point of **x** to obtain **x** + **y.** In figure 20.7(c), the initial point of **x** is placed on the terminal point of **y** to obtain **y + x.**

Exercise 20.1

Use figure 20.8 to answer questions 1, 2 and 3:

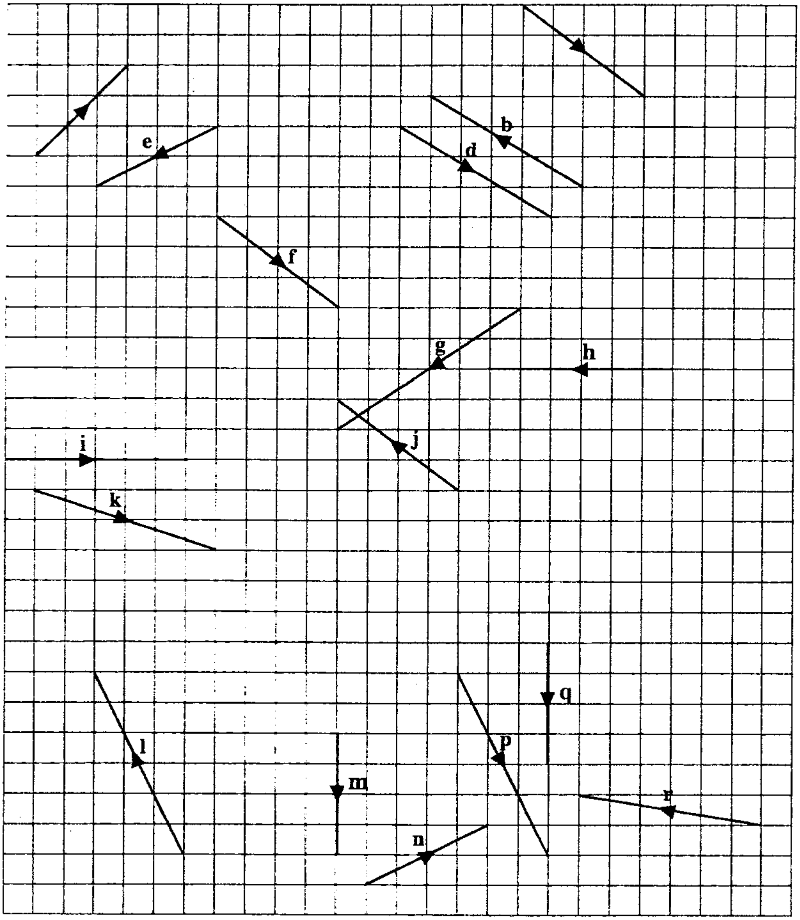


Fig. 20.8

1. Name pairs of equal vectors.
2. Name pairs of vectors with equal magnitudes but opposite directions.
3. Use graph paper to illustrate the sums of the following vectors:

   (a) **a** + **d** (b) **b** + **e** (c) **e** + **h**

   (d) **j** + **f** (e) **c** + **i** + **r** (f) **f** + **g** + **m**
4. ABCDEFGH is a cuboid with vectors **AF**, **AB** and **BC** given as **a**, **b** and **c** respectively, as in figure 20.9. Express all the diagonal vectors in terms of **a**, **b** and **c**.

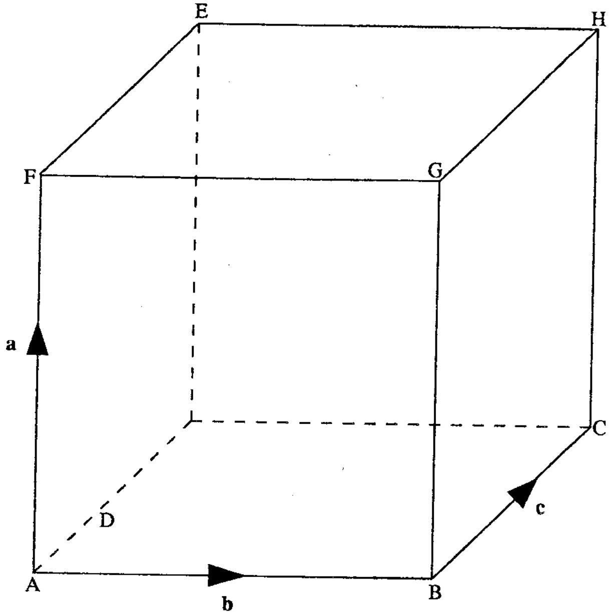


Fig. 20.9

The Zero Vector

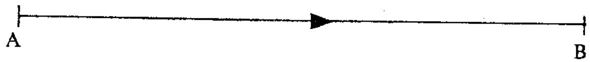


Fig. 20.10

Consider a displacement from A to B and back to A. The total displacement is zero. A zero displacement is represented by a vector of zero magnitude, denoted by **O**. This vector is called a **zero** or **null vector**. In this case, **AB + BA = O.**

The vector **BA** has the same magnitude as the vector **AB,** but its direction is opposite to that of **AB**, i.e., **BA** = –**AB.**

Generally, if **a** + **b** = 0, then;

b = –**a,** or, **a** = –**b**

20.5: Multiplication of a Vector by a Scalar

Positive Scalar

In figure 20.11, **AB** = **BC** = **CD** = **a.**

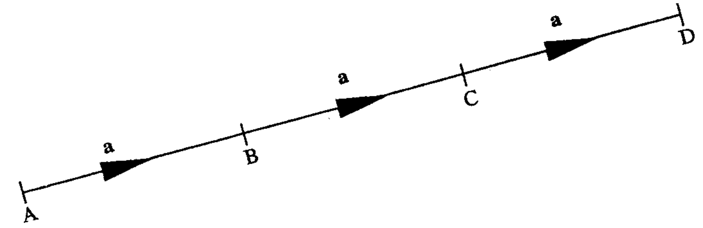


Fig. 20.11

Therefore, **AD** = **a** + **a** + **a** = 3**a**

This vector has the same direction as **a** and a magnitude three times that of **a**. In general, k**a**, where k is positive number, is a vector which has the same direction as **a** and magnitude k times that of **a.**

Negative Scalar

In figure 20.12, **DA** = (–**a**) + (–**a**) + (–**a**)
= –3**a**

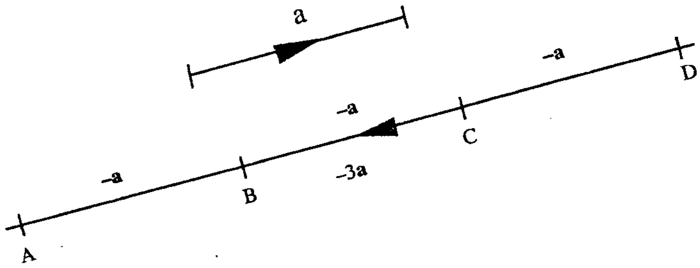


Fig. 20.12

The vector – 3**a** has magnitude three times that of **a** and its direction is opposite that of **a.**

The vector k**a** (k negative) has direction opposite that of **a** and its magnitude is k times that of **a**. The negative sign does not affect the magnitude. It only affects the sense or the direction of the vector.

The Zero Scalar

When a vector **a** is multiplied by 0, its magnitude is zero times that of **a**. The result is a zero vector.

a.0 = 0.a = 0

Example 1

Vectors **a** and **b** are given below. Use the grid to construct:

(a) **b** – **a**

(b) 3**a** – 2**b**

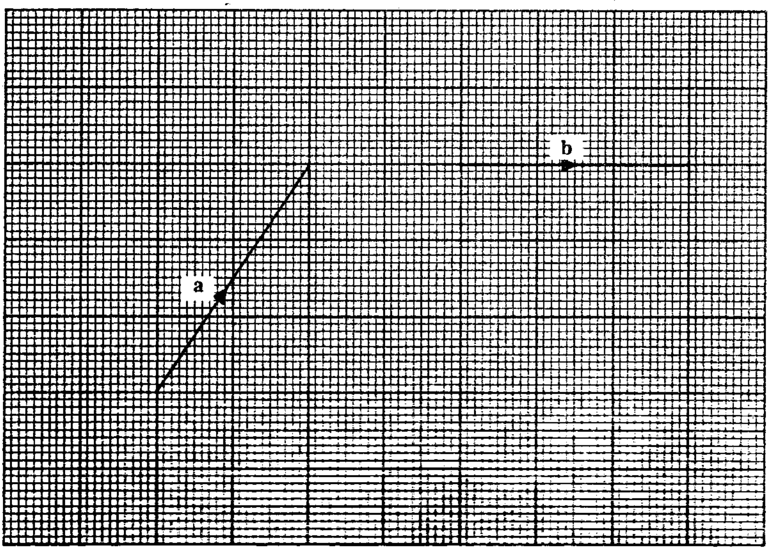


Fig. 20.13

Solution

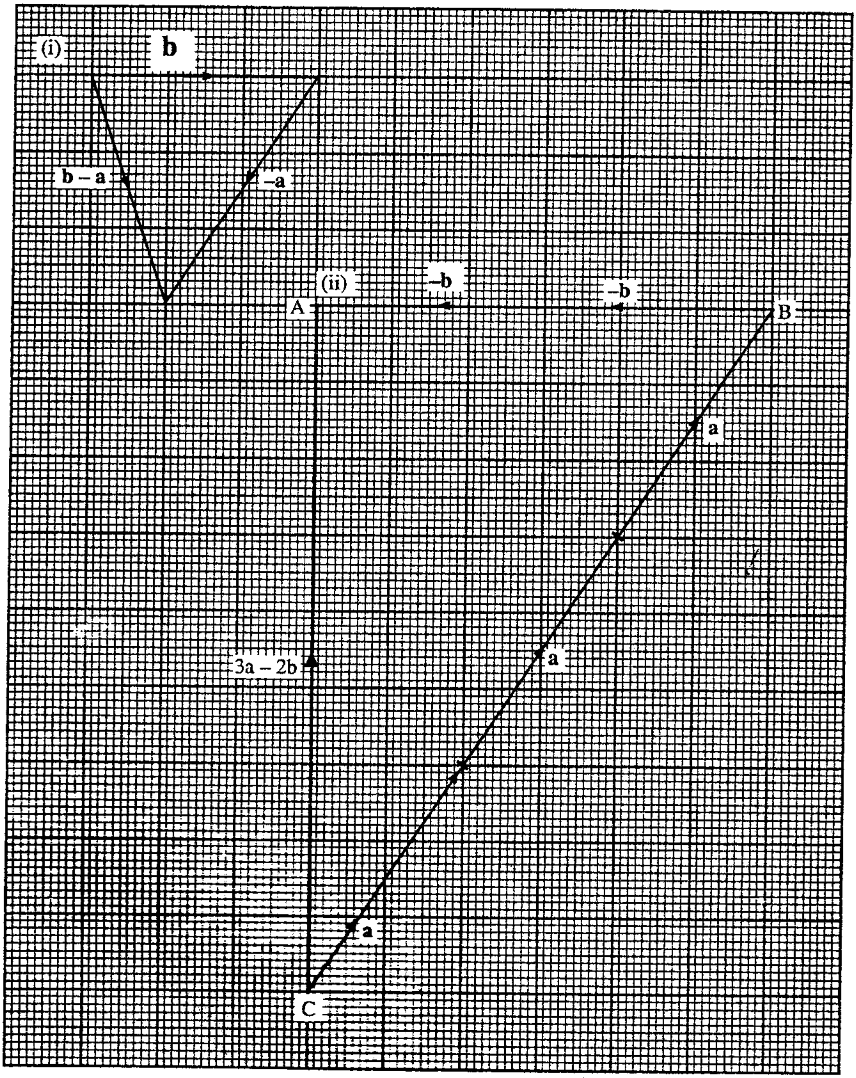


Fig. 20.14

Exercise 20.2

1. Figure 20.15 shows ABC a triangle in which the midpoints of AB, BC and AC are E, F and D respectively. Vector **AB** = –2**b** while **BC** = 2**a**.

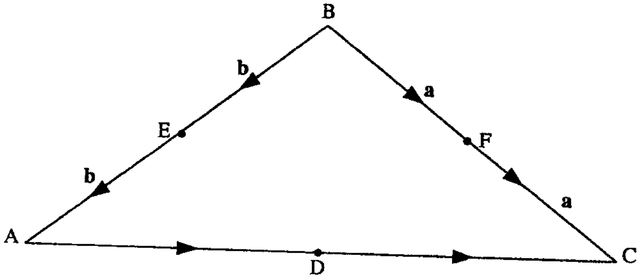


Fig. 20.15

Rewrite each of the following vectors in terms of **a** and **b**:

(a) **BF** (b) **AF** (c) **AC**
(d) **DC** (e) **DA** (f) **BD**

2. For each of the pairs of vectors shown in figure 20.16, write, where possible, the relationship between **p** and **q** in the form:
(i) **q** = m**p**,
(ii) **p** = n**q**,
where m and n are scalars.

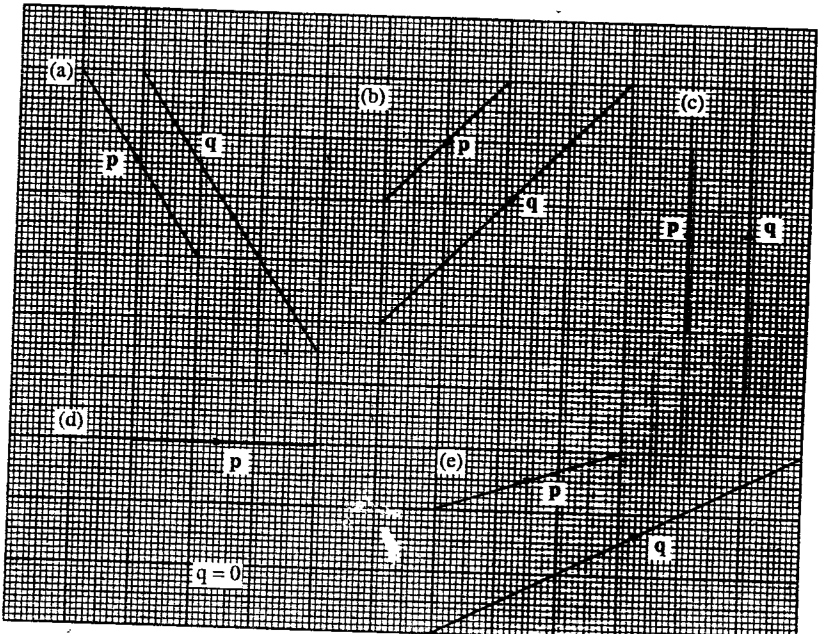


Fig. 20.16

3. Given that $\mathbf{t} = 2\mathbf{a} - \mathbf{b}$ and $\mathbf{r} = \mathbf{b} + 3\mathbf{a}$, find, in terms of **a** and **b**, the following vectors:

(a) $2\mathbf{t}$ (b) $\frac{1}{2}\mathbf{r}$ (c) $-\mathbf{t}$ (d) $-2\mathbf{r}$

(e) $2\mathbf{t} - 2\mathbf{r}$ (f) $-\mathbf{t} + \frac{1}{2}\mathbf{r}$ (g) $2\mathbf{r} - \mathbf{t}$

4. Copy vectors **a**, **b**, **c** and **d** in figure 20.17 on graph paper and use your copy to construct the following vectors:

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$ (c) $\mathbf{c} + \mathbf{d}$
(d) $\mathbf{c} - \mathbf{d}$ (e) $\mathbf{b} + \mathbf{c}$ (f) $\mathbf{b} - \mathbf{c}$
(g) $\mathbf{a} + \mathbf{c}$ (h) $\mathbf{a} - \mathbf{c}$ (i) $\mathbf{a} + \mathbf{d}$
(j) $\mathbf{a} - \mathbf{d}$

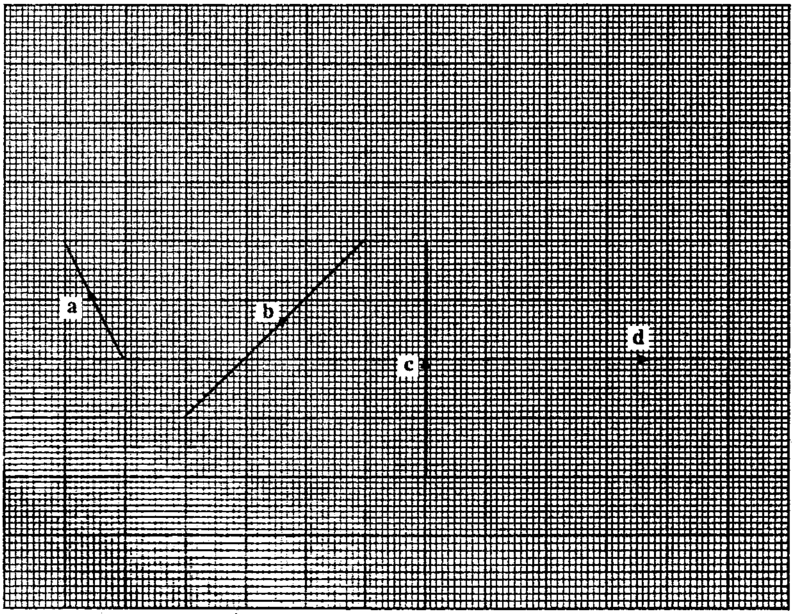


Fig. 20.17

5. Use vectors **a, b, c** and **d** in figure 20.17 to construct:

(a) $2\mathbf{a} - \frac{1}{3}\mathbf{b}$ (b) $\mathbf{b} - 2.5\mathbf{a}$ (c) $\frac{1}{3}\mathbf{b} + \frac{1}{4}\mathbf{c} - \mathbf{a}$
(d) $\mathbf{d} + \mathbf{c} - 2\mathbf{a}$ (e) $-2\mathbf{a} - \mathbf{b} - \mathbf{d}$

6. Figure 20.18 shows a regular hexagon PQRSTU. **PQ = a, QR = b** and **RS = c:**

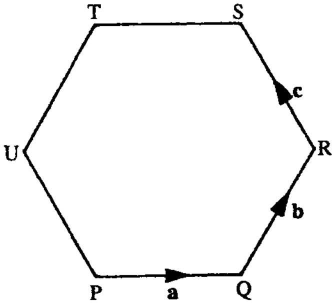


Fig. 20.18

Write in terms of **a**, **b** and **c** each of the following:
(a) **QR** (b) **PS** (c) **TQ** (d) **TP** (e) **RT** (f) **UQ** (g) **US**

7. Simplify:

   (a) $3(\mathbf{a}-2\mathbf{b})-4(5\mathbf{a}-\mathbf{b})$ (b) $\frac{1}{3}(\mathbf{a}-\mathbf{b})+\frac{1}{4}(3\mathbf{b}-4\mathbf{a})$

8. If $\frac{1}{3}\mathbf{a}+\frac{1}{5}\mathbf{b}=0$, write **a** as a multiple of **b**.
9. Three vectors **x**, **y** and **z** satisfy the two equations $\mathbf{x}+\mathbf{y}+\mathbf{z}=0$ and $2\mathbf{x}-\mathbf{y}-3\mathbf{z}=0$. Express the vectors **y** and **z** in terms of **x** only.
10. Simplify:

    (a) $3\mathbf{a}+2\mathbf{b}-\mathbf{c}+4(\frac{1}{2}\mathbf{a}-\frac{3}{2}\mathbf{c})+\frac{1}{3}(6\mathbf{a}-9\mathbf{b})$

    (b) $2\mathbf{u}-3\mathbf{v}+2(\mathbf{w}-\mathbf{u})+3(\mathbf{u}+\mathbf{v})$

    (c) $(\mathbf{p}-\mathbf{q})+(\mathbf{r}-\mathbf{p})+(\mathbf{q}-\mathbf{r})$

11. Figure 20.19 shows a large cube divided into 27 smaller ones. $\mathbf{AB}=\frac{3}{2}\mathbf{a}$, $\mathbf{BC}=3\mathbf{c}$ and $\mathbf{BE}=\mathbf{b}$:

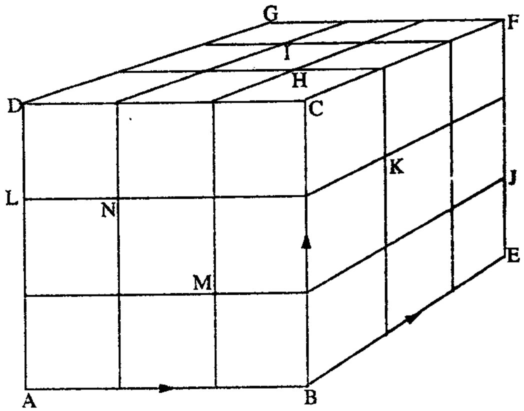


Fig. 20.19

Write each of the following in terms of **a**, **b** and **c**:

(a) **AE** (b) **BD** (c) **BF** (d) **AG** (e) **AJ** (f) **AK**
(g) **AI** (h) **MN** (i) **LN** (j) **NJ** (k) **MI** (l) **2NL + HD**
(m) **MK + KI + IM** (n) **JN − NM** (p) **NK + IF** (q) **3MK − 2NH**

20.6: Column Vectors

Consider points A(2, 1) and B(4, 7) in figure 20.20. The vector **AB** represents a displacement from A(2, 1) to B (4, 7). This displacement consists of two components **AE** and **EB**.

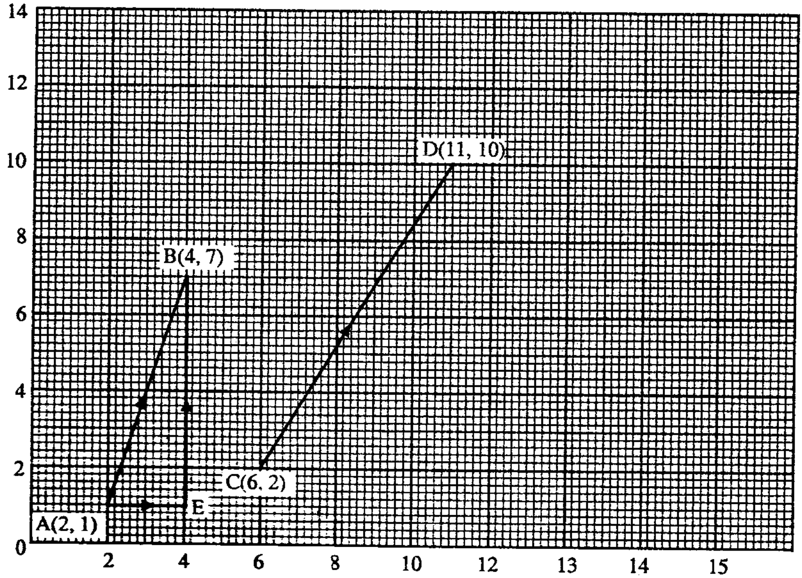


Fig. 20.20

The components are 2 units $(4-2)$ in the horizontal direction and 6 units $(7-1)$ in the vertical direction. We represent vector **AB** in terms of its components as $\begin{pmatrix}2\\6\end{pmatrix}$.

Vectors of the form $\begin{pmatrix}a\\b\end{pmatrix}$ where **a** is the horizontal displacement and b the vertical displacement are known as **column vectors**. What is the column vector representing vector **CD** in the diagram?

Note:

(i) Displacements parallel to the x-axis are positive if they are to the right and negative if they are to the left.

(ii) Displacements parallel to the y-axis are positive if they are upwards and negative if they are downwards.

Example 2

Find the sum of each of the following column vectors and illustrate their solutions graphically:

(a) $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$ (b) $\mathbf{c} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Solution

(a) Total displacement in x direction is $2 + 7 = 9$.
Total displacement in y direction is $3 + 1 = 4$.

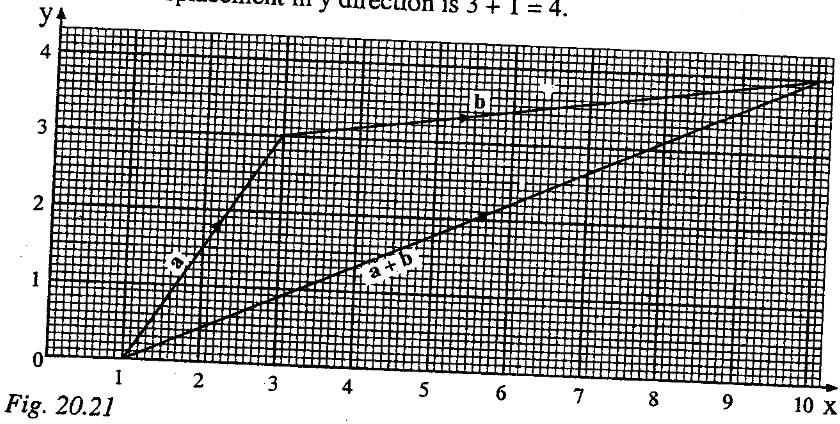


Fig. 20.21

Therefore, $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

(b) Total displacement in x direction is $(-4 + 5) = 1$
Total displacement in y direction is $(2 + -2) = 0$

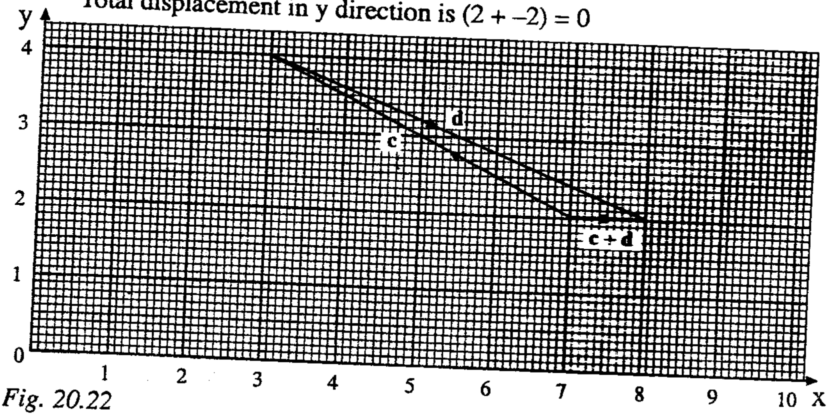


Fig. 20.22

Therefore, $\mathbf{c} + \mathbf{d} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Example 3

If $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, find $3\mathbf{a} + 2\mathbf{b}$.

Solution

$3\mathbf{a} = 3\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

$2\mathbf{b} = 2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

Therefore, $3\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$

20.7: The Position Vector

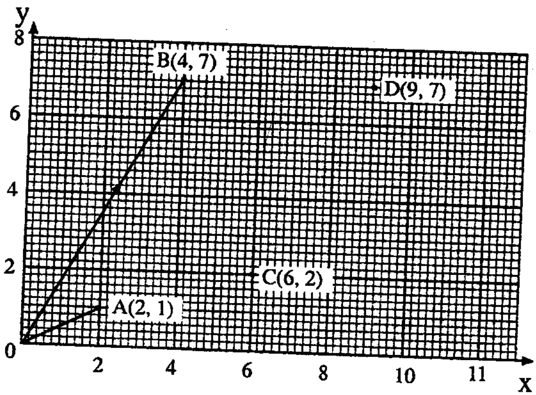


Fig. 20.23

Each of the points A(2, 1) and B(4, 7) has been located in the plane relative to the origin O. **OA** is known as the position vector of A relative to the origin.

$$\mathbf{OA} = \begin{pmatrix} 2-0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Similarly, the position vector of B is $\mathbf{OB} = \begin{pmatrix} 4-0 \\ 7-0 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

What are the position vectors of points C and D in figure 20.23?

For any point A in the plane, its position vector **OA** is denoted by **a**. In the same way, the position vector **OB** is denoted by **b**.

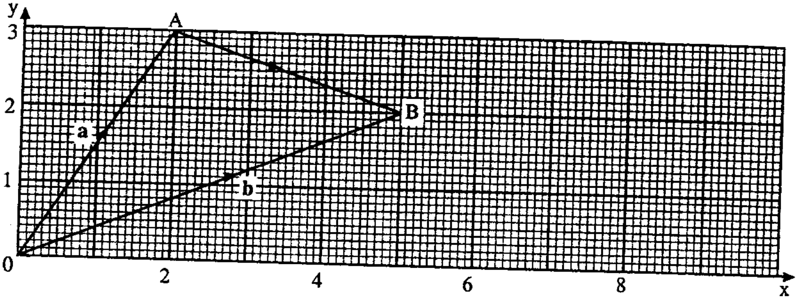


Fig. 20.24

In the above figure, $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

The column vector **AB** $= \mathbf{b} - \mathbf{a}$

$$= \begin{pmatrix} 5-2 \\ 2-3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

In general, if **a** is the position vector of A and **b** the position vector of B, then the vector **AB** = **b** – **a**.

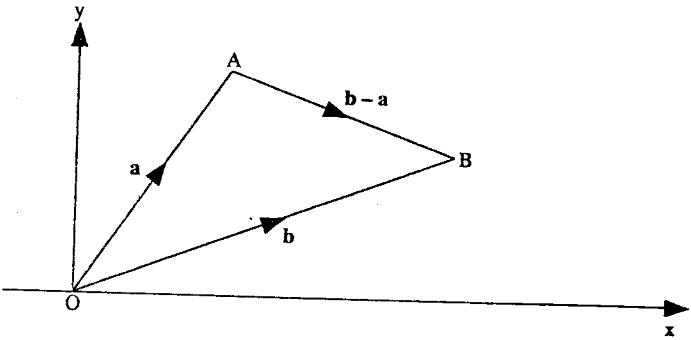


Fig. 20.25

In figure 20.25, **a** + **AB** = **b**. Thus, **AB** = **b** – **a**.

Exercise 20.3

1. Write each of the vectors below as a column vector:

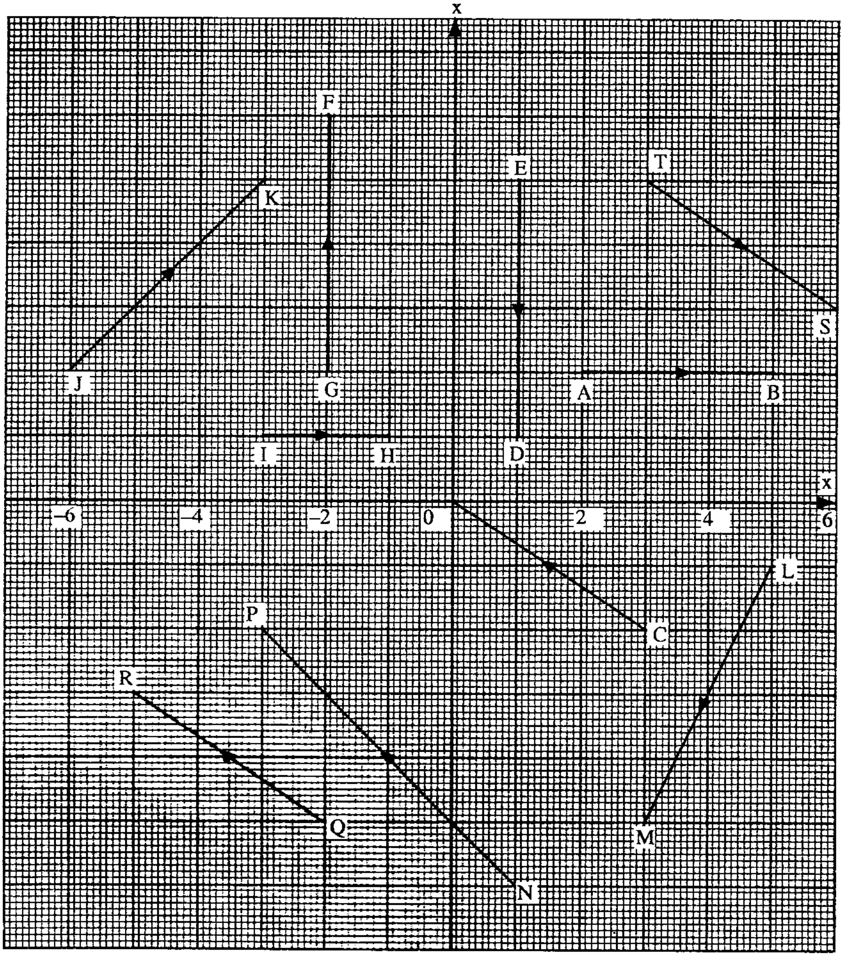


Fig. 20.26

2. Draw the following position vectors on squared paper:

$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\mathbf{e} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\mathbf{f} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ $\mathbf{g} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ $\mathbf{h} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3 Use figure 20.26 to write down the position vectors of points E, C, Q, L, P and I.

4. If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$, find:

(a) (i) $3\mathbf{a} - 2\mathbf{b}$ (ii) $5\mathbf{b} - 3\mathbf{a}$ (iii) $\frac{1}{2}\mathbf{c} - \frac{3}{2}\mathbf{d}$

(iv) $3\mathbf{b} + \mathbf{c}$ (v) $\frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$ (vi) $5\mathbf{b} + 7\mathbf{d}$

(vii) $10\mathbf{a} + 15\mathbf{b} + 23\mathbf{d}$

(b) the scalars r and s such that $r\mathbf{a} + s\mathbf{c} = 9\mathbf{b}$.

5. If $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, find $-\mathbf{y}$, given that $\mathbf{x} + \mathbf{y} = \mathbf{z}$.

6. Find the co-ordinates of P if $\mathbf{OP} = \mathbf{OA} + \mathbf{OB} - \mathbf{OC}$ and the co-ordinates of points A, B and C are (3, 4), (–3, 4) and (–3, –4) respectively.

7. Find scalars λ and μ such that $\lambda\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

20.8: Magnitude of a Vector

Figure 20.27 shows a right-angled triangle ABC. $AB = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

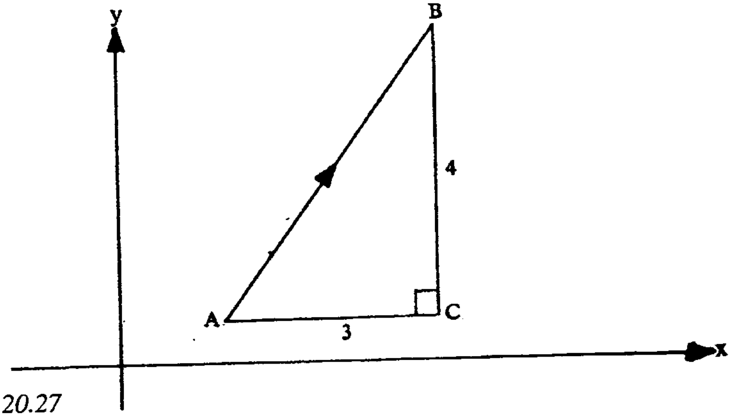


Fig. 20.27

Using Pythagoras' theorem, the length of line AB is:

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Hence, the magnitude of vector **AB** is 5, This can also be written as $|\mathbf{AB}| = 5$.

In general, if $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the length or the magnitude of the vector **a** is given by $\sqrt{x^2 + y^2}$.

Calculate the magnitude of each of the following vectors:

(i) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 8 \\ 15 \end{pmatrix}$

20.9: Midpoints

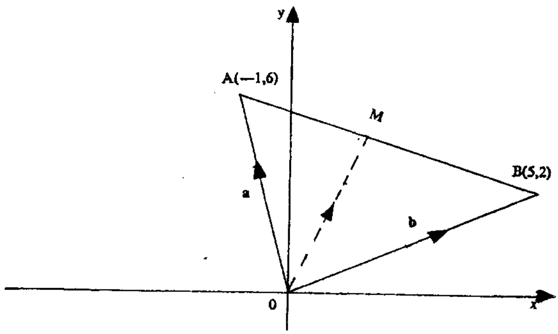


Fig. 20.28

In figure 20.28, M is the midpoint of AB. We can use vector method to get the co-ordinates of M.

$$\begin{aligned}
\mathbf{OM} &= \mathbf{OA} + \left(\tfrac{1}{2}\right)\mathbf{AB} \\
&= \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \tfrac{1}{2}\left[\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \end{pmatrix}\right] \\
&= \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 6 \\ -4 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} -1+3 \\ 6-2 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 4 \end{pmatrix}
\end{aligned}$$

Therefore, the co-ordinates of M are (2, 4).

Note:

Co-ordinates of a point must be given in horizontal form.

Consider the general case in figure 20.29, where M is the midpoint of AB.

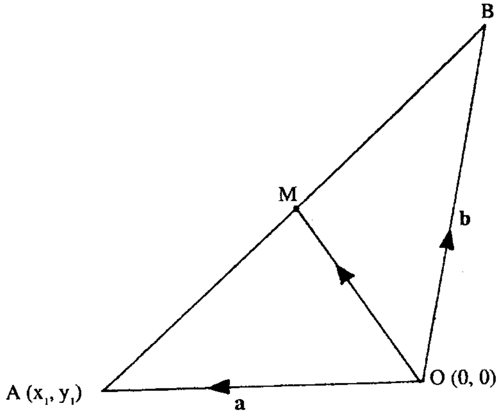


Fig. 20.29

$$\begin{aligned}\mathbf{OM} &= \mathbf{OA} + \mathbf{AM}\\ &= \mathbf{a} + \tfrac{1}{2}(\mathbf{AB})\\ &= \mathrm{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a})\\ &= \mathrm{a} + \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}\mathbf{a}\\ &= \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}\\ &= \frac{\mathbf{a}+\mathbf{b}}{2}\end{aligned}$$

But $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Therefore, $\dfrac{\mathbf{a}+\mathbf{b}}{2} = \dfrac{1}{2}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

Then, M is the point $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

For example, in figure 20.28, since M is the midpoint, the co-ordinates of M are

$\left(\dfrac{-1+5}{2}, \dfrac{6+2}{2}\right)$, which is (2, 4).

Exercise 20.4

1. Find the coordinates of the midpoint of **AB** in each of the following cases:
   (a) A(–4, 3), B(2, 0) (b) A(0, 4), B(0, –2)
   (c) A(–3, 2), B(4, 2) (d) A(0, 0), B(a, b)
2. In figure 20.30, OAB is a triangle. A is the point (2, 8) and B the point (10, 2). C, D and E are the midpoints of **OA, OB and AB** respectively.

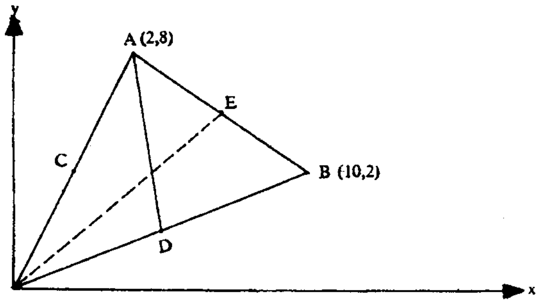


Fig. 20.30

(a) Find the co-ordinates of C and D.

(b) Find the length of the vectors **CD** and **AB**.

3. Triangle OAB is such that **OA = a and OB = b.** C lies on **OB** such that **OC** : **CB** = 1 : 1. D lies on **AB** such that **AD** : **DB**= 1 : 1 and E lies on **OA** such that **OA**: **AE** = 3 : 1. Find:

(a) **OC** (b) **OD** (c) **OE** (d) **CD** (e) **DE**

20.10: Translation

If an object is moved in such a way that all its points move the same distance and in the same direction, the object is said to have undergone a **translation**. In figure 20.31, Δ A′B′C′ is the image of ΔABCunder a translation.

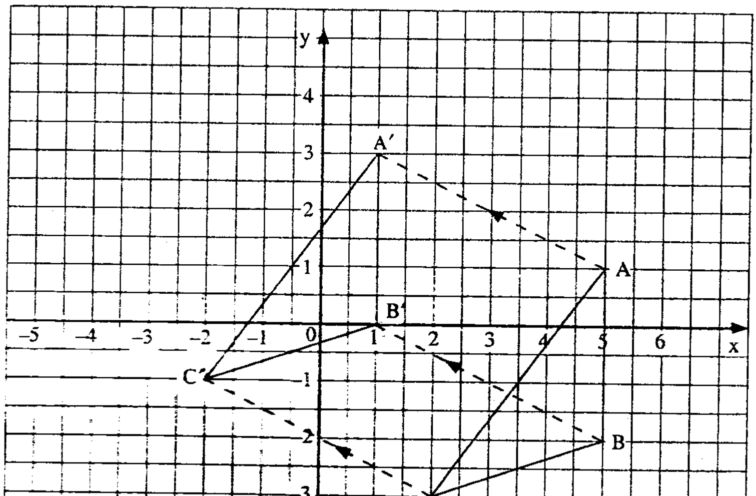


Fig. 20.31

Note:

(i) All points on ΔABC move equal distance in the same direction. This translation is described by the vector $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$.

(ii) Under a translation, an object and its image are directly congruent. If letter **T** is used to denote a translation, then **T**(P) represents translation **T** on P.

For example, if **T** is translation $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and P is the point (3, 1), we can get P', the image of P under **T**(P), as follows:

The position vector of P is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

The position vector of P', the image of P, is;

$$OP' = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

Therefore, the co-ordinates of P' are (8, −1).

Example 4

The points A(−4, 4), B(−2, 3), C(−4, 1) and D(−5, 3) are vertices of a quadrilateral. If the quadrilateral is given the translation **T** defined by the ~~vector~~ $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$, draw the quadrilateral ABCD and its image under **T**.

Solution

$OA' = \begin{pmatrix} -4 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so A' is (1, 1).

$OB' = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, so B' is (3, 0)

$OC' = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, so C' is (1, −2)

$OD' = \begin{pmatrix} -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; so D' is (0, 0).

Quadrilaterals ABCD and A′B′C′D′ are shown in figure 20.32.

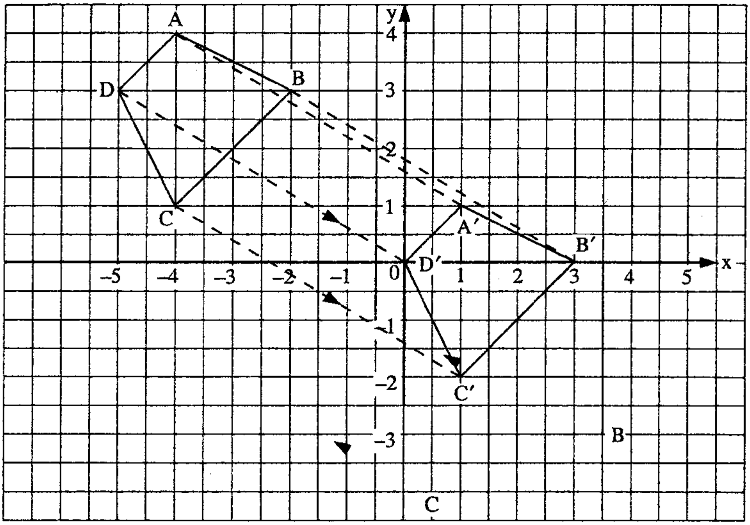


Fig. 20.32

Exercise 20.5

1. The co-ordinates of A, B, C and D are (–2, 5), (1, 3), (3, –2) and 2, –4) respectively. If A' is (–5, 6) under a translation **T**, find the co-ordinates of B', C' and D' under **T**.
2. Draw ΔPQR with vertices P(3, 2), Q(5, 0) and R(4, –1). On the same axes, plot $P'Q'R'$, the image of ΔPQR under a translation given by $\begin{pmatrix}-4\\1\end{pmatrix}$.
3. A point P(–2, 3) is given the translation $\begin{pmatrix}-1\\4\end{pmatrix}$. Find the point P', the image of P under the translation. If P' is given a translation $\begin{pmatrix}1\\-2\end{pmatrix}$, find the co-ordinates of P'', the image of P'. What single translation maps P onto P''?
4. The point A(3, 2) maps onto A' (7, 1) under a translation T_1. Find T_1. If A' is mapped onto A'' under translation T_2 given by $\begin{pmatrix}-3\\5\end{pmatrix}$, find the co-ordinates of A''. Given that $T_3(A) = A''$, find T_3.
5. For each of the points below, give the image when it is translated by the given vector:

   (a) (2 , 3); $\begin{pmatrix}4\\3\end{pmatrix}$ (b) (0, 0); $\begin{pmatrix}-2\\1\end{pmatrix}$ (c) (–2, –4); $\begin{pmatrix}4\\3\end{pmatrix}$

(d) $(-4, 3)$; $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (e) $(7, 0)$; $\begin{pmatrix} -8 \\ -1 \end{pmatrix}$ (f) $(8, -2)$; $\begin{pmatrix} -10 \\ 4 \end{pmatrix}$

6. Below are images of points which have been translated by the vectors indicated. Find the corresponding object points:

(a) $(2, 3)$; $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (b) $(0, 0)$; $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

(c) $(9, 17)$; $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (d) $(5, 6)$; $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$

(e) $(7, -2)$; $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ (f) $(-2, 4)$; $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$

(g) $(3, 3)$; $\begin{pmatrix} -4 \\ -10 \end{pmatrix}$ (h) $(-5, -11)$; $\begin{pmatrix} -12 \\ -0 \end{pmatrix}$

(i) $(4, 3)$; $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (j) (a, b); $\begin{pmatrix} c \\ d \end{pmatrix}$

7. The vertices of a triangle are A(5, 1), B(4, 4) and C(2, 1). Draw the image of the triangle ABC after a translation given by the vector .

8. The image of point (6, 4) is (3, 4) under a translation. Find the translation vector.

9. In figure 20.33, A′, B′, C′, D′, E′, F′ and G′ and are the images of A, B, C, D, E, F, G under a translation. Find the translation vector.

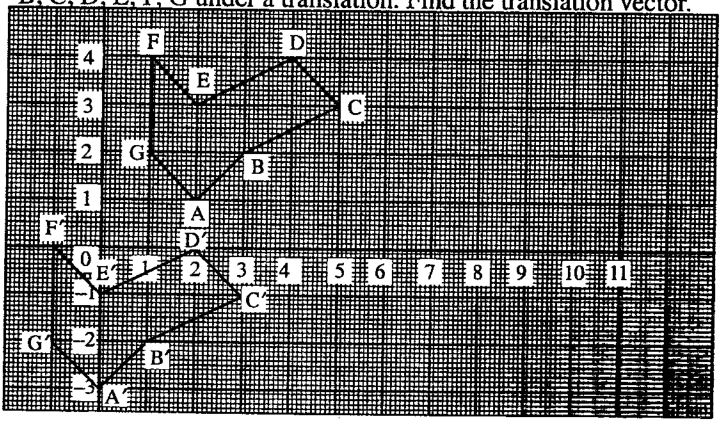


Fig. 20.33

10. (a) Use squared paper to draw the images of PQRS in figure 20.34 below under the following translation vectors:

(i) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ (iii) $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ (iv) $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$

(b) Find the translation vector which would map your image in (a) (ii) to the image in (a) (iv) above.

(b) Find the translation vector which would map the image in (a)(iv) to that in (a)(i).

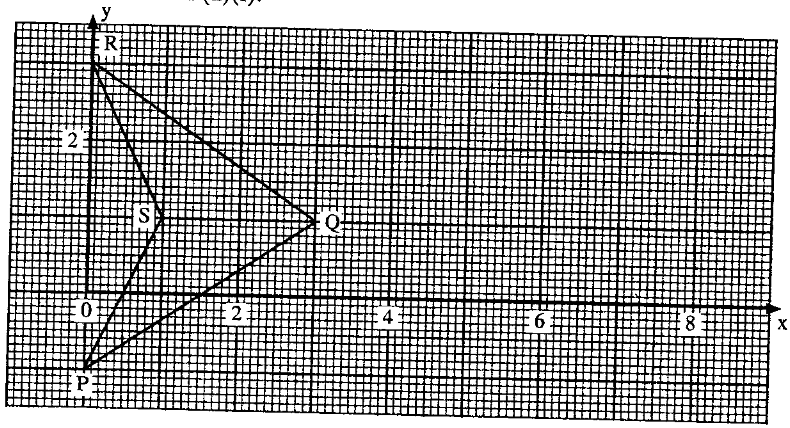


Fig. 20.34

Mixed Exercise 3

1. A particle moves from rest and attains a velocity of 10 m/s after two seconds. It then moves with this velocity for four seconds. It finally decelerates uniformly and comes to rest after another six seconds.
   (a) Draw a velocity – time graph for the motion of this particle.
   (b) From your graph find:
      (i) the acceleration during the first two seconds.
      (ii) the uniform deceleration during the last six seconds.
      (iii) the total distance covered by the particle.
2. The mean of the numbers m, 8m + 1, 17 and 20 is 14. Calculate:
   (a) the value of m.
   (b) the mode.
3. Ten casual labourers were hired by a garment factory for one week and paid in shillings, according to productivity of each, as below:
   615, 633, 720, 509, 633, 710, 614, 630, 633, 720
   Find the mean, mode and median for the data.
4. In a bakery, the mass of 20 loaves taken at random is 10.03 kg. If the mean mass of the first 13 loaves is 505 g, find the mean mass of the other 7 loaves.
5. The total marks scored in a test by 6 pupils was 420. If the mean mark for the first 5 pupils was 68, find the marks scored by the sixth pupil.
6. At a police check point, the speeds in km/h of the first 50 vehicles were recorded as follows:

Speed	Number of vehicles
10 – 19	3
20 – 29	1
30 – 39	2
40 – 49	5
50 – 59	6
60 – 69	11
70 – 79	9
80 – 89	8
90 – 99	3
100 – 109	2

   (a) Calculate the mean speed.
   (b) Draw a histogram and estimate the median.
   (c) On the same diagram, draw a frequency polygon.

7. The table below shows a record of the number of goals scored in various matches by football team:

No. of goals	No. of matches
0	8
1	5
2	6
3	5
4	5
5	4
6	3

Calculate:
(a) the mean number of goals per match.
(b) the mean number of matches per goal.

8. Maina divided his farm among his four sons. The first son got twice as much as the second. The third got 2 hectares more than the first, while the fourth got 2 hectares less than the second. If on a pie chart, the share of the fourth son is represented by a sector of 48^0, find the size of the shamba.

9. The heights of 40 pupils in a class were measured to the nearest centimetre and recorded as below:

175	154	157	180	165	150	152	162
173	168	169	181	177	179	175	169
151	153	156	158	163	169	179	180
145	149	150	156	171	175	176	178
169	160	155	174	170	176	182	170

(a) Use class intervals of 5 to group the data.
(b) Estimate the mean and the median height of the class.

10. The mean of a set of n numbers is 28. If an extra number, 18, is included in the set, the mean now becomes 26. Calculate the value of n.

11. The mean of 100 numbers is 13. If each of these is multiplied by 3 and 5 subtracted from the result, calculate the new mean.

12. The height of 25 coffee shrubs in a certain farm in centimetres were recorded as below:

109	120	114	121	150
149	125	106	148	140
109	108	130	138	142

148	147	107	136	139
132	135	109	116	128

Illustrate this information on a frequency table using a class interval of 5 cm.

(a) What is the modal class?

(b) Calculate the mean height.

13. Solve the equation ;

$\frac{1}{5}x^2 + 3x = -10$

14. The co-ordinates of points A, B and C are (O, –4), (2, –1) and (4, 2) respectively.

(a) Deduce the position vectors of A, B and C.

(b) Find the lengths of **AB** and **AC**.

(c) Show that the points A, B and C are collinear.

15. The points P(–2, 1), Q(1, 4) and R(3, 1) are vertices of a triangle PQR. If the triangle is given a translation **T** defined by the vector $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$, draw the triangle PQR and its image.

16. A pentagon is described by the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} -3 \\ 11 \end{pmatrix}$ and **e** in that order. What is the relationship between the vectors **c** and **e**?

17. Solve the equation $x^2 = 4(\frac{1}{2}x + 2)$

18. Simplify:

(a) $\dfrac{49a^2 - 9b^2}{14a + 7ab + 6b + 3b^2}$ (b) $\dfrac{(4x + 2m)^2(-4m^2 + 16x^2)}{(4m^2 + 16x^2 + 16mx)(-2m + 4x)}$

(c) $\dfrac{r^2 - k^2 + r + k}{r - k + 1}$

19. Use vectors to show that the points A(6, 1), B(3, 4) and C(1, 6) lie on a straight line. If O is the origin and E and F are the midpoints of **OC** and **OA** respectively, find the co-ordinates of E and F . Show that **AC** || **EF**. If OAC is reflected in the x-axis, show that:

(a) $|\mathbf{AC}| = |\mathbf{A'C'}|$

(b) $|\mathbf{EF}| = |\mathbf{E'F'}|$

20. The column vectors of two sides of a triangle PQR are given by $\mathbf{PQ} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\mathbf{QR} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$. Write down the column vectors **RP** and **PR**. Determine

the vectors representing the sides of the image of ΔPQR after a reflection:

(a) in the x axis.

(b) in the y axis.

(c) in the straight line $y = 2x$.

21. Find possible values of x if $9^{x^2} = 27^{(2x+12)}$

22. The point P(–2, 5) is mapped onto $P'(1, 9)$ by a translation $\mathbf{T}_1$. If P' is mapped onto P'' by a translation $\mathbf{T}_2$ given by $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$, find the co-ordinates of P''. Given that $\mathbf{T}_3(P) = P''$ Find $\mathbf{T}_3$.

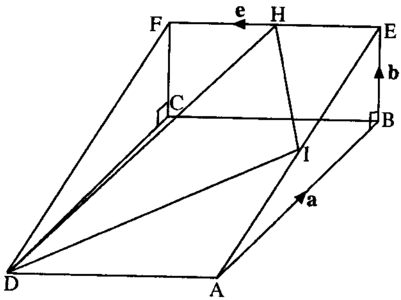
23. If $\mathbf{a}$ $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 14 \\ 1 \end{pmatrix}$, find:

(a) $2\mathbf{a} - 3\mathbf{b}$.

(b) $\frac{1}{2}\mathbf{a} - \mathbf{b}$.

(c) scalars x and y if $x\mathbf{a} - y\mathbf{b} = \begin{pmatrix} 16 \\ 11 \end{pmatrix}$.

24. In the figure below, ABCD is a parallelogram and CBEF is a rectangle. H and I are midpoints of **EF** and **AE** respectively. **AB** = **a**, **BE** = **b** and **EF** = **e**.



Write the following vectors in terms of **a**, **b** and **e**:

(a) **DA** (b) **AE** (c) **AF** (d) **CE** (e) **DH** (f) **DI** (g) **IH**

25. Factorise the expressions:

(a) $(2x + 2y)^2 - (x - y)^2$ (b) $x^4 - (2x - 3)^2$

(c) $(2x + y)^2 - (x - y)^2$ (d) $x^4 - (2x + 3)^2$

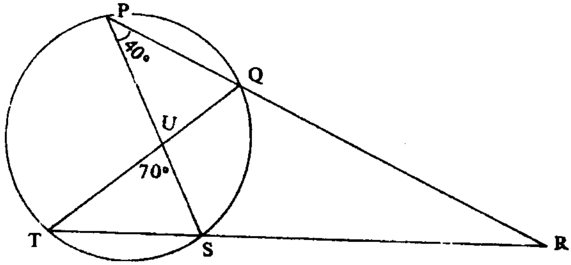
26. A triangle with vertices at P(–1, 3), Q(–3, 5) and R(–2, 8) is reflected in the line $y = 3x - 1$ to get triangle $P'Q'R'$. Triangle $P'Q'R'$ is then translated by vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ to get $\Delta P''Q''R''$. Draw triangles PQR, $P'Q'R'$ and $P''Q''R''$.

27. Solve the simultaneous equations;

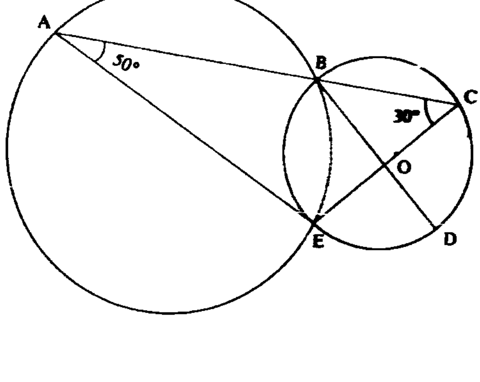
$$2\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$-\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

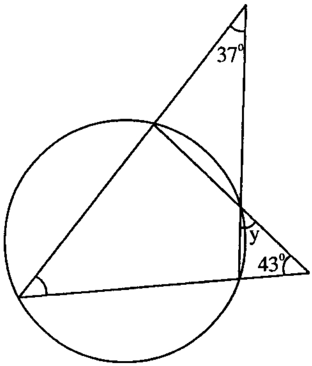
28. The dimensions, of a rectangle in centimetres are $2n - 3$ by $n + 1$ and the area is 817 cm^2. Determine the length of the diagonal.
29. A man subdivides his land of 28 hectares into two pieces, one square and the other rectangular. The length of the rectangular piece is 600 m and its width is half the length of the square. Which piece is larger and by how much?
30. In the figure below, $\angle SPQ = 40^0$ and $\angle TUS = 70^0$. Calculate $\angle PRT$:



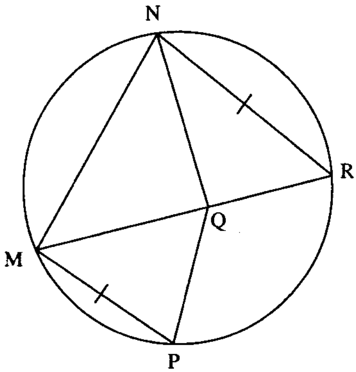
31. In quadrilateral ABCD, $\angle ADC = 115^0$, $\angle ACB = 40^0$, $\angle DBC = 35^0$ and $\angle BAC = 75^0$. Show that A, B, C, D are concylic. Hence, find:
(a) $\angle DAC$.
(b) $\angle DCA$.
(c) $\angle BDC$.
32. In the figure below, O is the centre of circle BCDE, $\angle BAE = 50^0$ and $\angle BCE = 30^0$. Find:
(a) $\angle AEB$.
(b) $\angle BDC$.



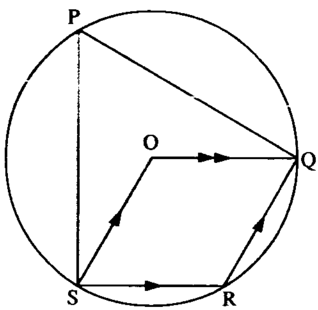
33. In the figure below calculate the value of y:



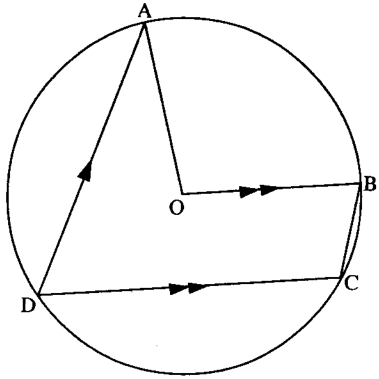
34. In the figure below, $\angle RMN = 40^\circ$, $\angle QNR = 30^\circ$, MP = NR and MQR is a straight line passing through the centre of the circle. Find $\angle NRM$ and $\angle QNP$.



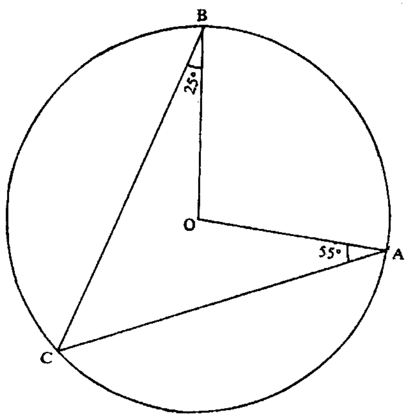
35. The figure below shows a circle PQRS with centre O. If OQ is parallel to SR and SO is parallel to RQ. show that $\angle SPQ = \angle RSO$:



36. The figure below shows a circle ABCD, centre O. Show that $\angle AOC = 2\angle OBC$:



37. In the figure below, O is the centre of circle ABC, $\angle OAC = 55°$ and $\angle OBC = 25°$. Calculate the reflex $\angle AOB$:



38. Illustrate on a number line the solutions to the following inequalities:

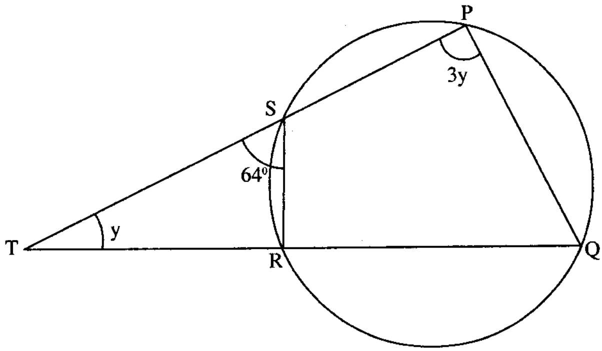
(a) (i) $x^2 < x$ (ii) $x^2 > x$

(b) (i) $2x < x$ (ii) $2x > x$

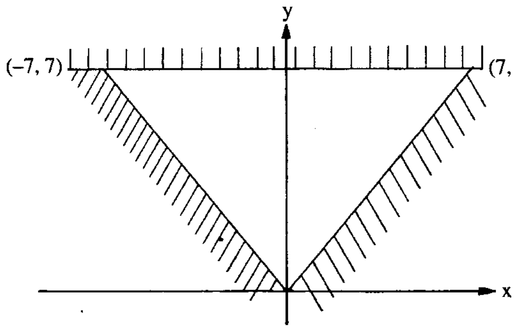
(c) (i) $\sqrt{x} > x$ (ii) $\sqrt{x} < x$

39. The region determined by the inequalities $x + y > 0$, $y \geq x$ and $2y - x \leq 6$ is reflected in the x-axis. Find the three inequalities that satisfy the image.

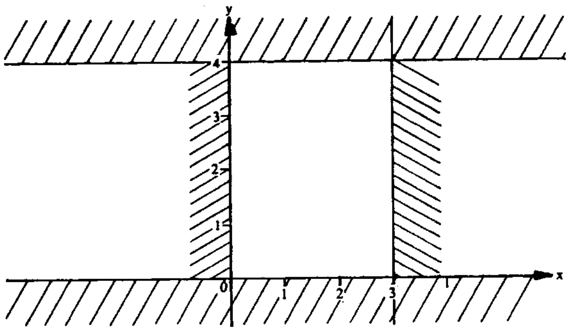
40. In the figure below, calculate the value of y and give reasons for your answer.



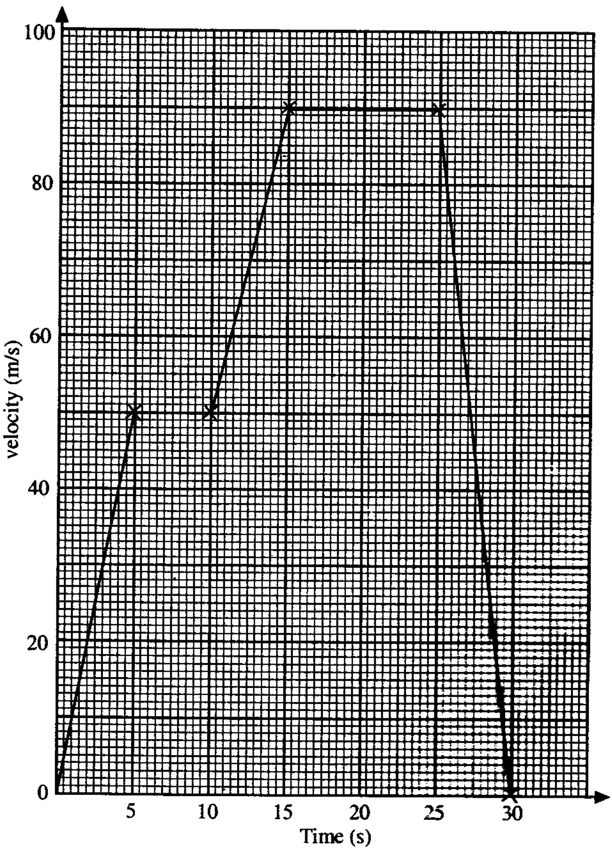
41. Represent the simultaneous inequalities $\frac{y}{3} + \frac{3}{8} \geq \frac{y}{4}$ and $\frac{y}{2} - 6 < \frac{y}{5}$ on a number line.
42. Represent on graph the region which satisfies the four inequalities; $y \geq -1$, $y \leq x - 1$, $y \leq -x + 9$ and $y + 12 \geq 2x$. Determine the area of the region.
43. Find the possible values of x + y if $3^{(x+y)^2} - 4 = 27^{(x+y)}$. Hence, find the possible values of x if $2^{x-y} = 1$
44. The unshaded region in the figure below is reflected in the x-axis. Write down the inequalities which satisfy the new region. Find the area of the region.



45. Represent the following inequalities on graph paper by shading out the unwanted region:
$y \leq 2x + 1$, $4y + x \leq 22$, $2y \geq x - 5$
If the straight line through the points (5, 2) and (6, 4) forms the fourth boundary of the enclosed region, write down the fourth inequality by the region (points on the line are in the region) .
46. Write down the inequalities satisfied by the illustrated region in the figure below:



47. The following is a velocity-time graph of a moving object:



Use the graph to calculate:

(a) the velocity during the first five seconds.

(b) distance covered in the first eight seconds

(c) total distance travelled.

(d) the acceleration during the 8^{th} and the 12^{th} seconds.

(e) the acceleration between the 14^{th} and 20^{th} seconds.

48. Find seven pairs of integral co-ordinates that satisfy all these inequalities:
$y + 3x \geq 5$
$3y + x \leq 15$
$y - x \geq 1$
Find the area and the perimeter of the triangular region bounded by these inequalities.

49. Find the area of the trapezoidal region bounded by the inequalities:
$y - 2x \leq 1$
$3y - x > -2$
$2y + x < 7$
$2y + x \geq 2$

50. A car which accelerates uniformly from rest attains a velocity of 30 m/s after 15 seconds. It travels at a constant velocity for the next 30 seconds before decelerating to stop after another 10 seconds. Use a suitable scale to draw the graph of velocity against time and use it to determine:

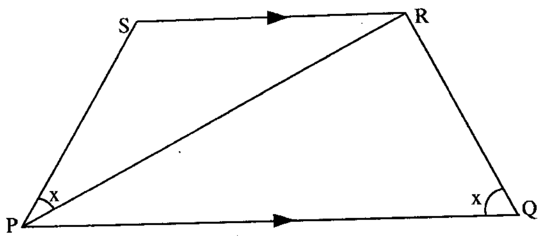
(a) the total distance covered by the car in the first 55 seconds.

(b) the retardation.

REVISION EXERCISES

Revision Exercise 1

1. Evaluate $\sqrt[4]{\dfrac{3.45 + 2.62}{786 \times 0.0007}}$.
2. The vertices of a kite are A(0, 0), B(5, 2), C(9, 9) and D (2, 5). Find the equation of the line of symmetry of the kite. If ABCD is reflected in the line x = 0, find:
   (a) the co-ordinates of the vertices of the image.
   (b) the equation of the line of symmetry of the image.
3. In the figure below, PQRS is a trapezium and $\angle SPR = \angle PQR = x$:



   (a) Show that triangles SPR and PRQ are similar.
   (b) If PQ = 18 cm and RS = 8 cm, find PR.
4. If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find:

   (a) $2\mathbf{a} + \frac{1}{2}\mathbf{b}$.
   (b) $3\mathbf{a} - 5\mathbf{b}$.
   (c) $\frac{7}{8}\left(3\mathbf{a} + \frac{1}{4}\mathbf{b}\right)$.
5. A right pyramid has a square base of sides 12 cm and slant height of 20 cm. Calculate:
   (a) its total surface area.
   (c) its volume.
6. Represent the following inequalities on a number line:

   (a) $3x + 7 < -5$ (b) $\frac{t}{3} \leq \frac{5}{6}t - 4$
   (c) $5(2r - 1) \leq 7(r + 1)$ (d) $6(x - 4) - 4(x - 1) \geq 52$
   (e) $1\frac{3}{4}k - 1 \leq 3k - 2 \times 6\left(\frac{2}{3}k + 2\right)$
7. The vertices of a quadrilateral are A(5, 1), B(4, 4), C(1, 5) and D(2, 2).

(a) Use gradients to show that:
(i) AB is parallel to CD. (ii) AD is parallel to CB.
(iii) AC is perpendicular to DB.

8. The amount of milk produced by 20 cows per day, to the nearest litre, was recorded as below:

No. of cows	2	2	4	3	4	5
Amount of milk in litres	8	10	7	9	6	5

(a) Find the mean number of litres produced by the cows.
(b) What is the mode of the data?
(c) Draw a histogram to represent the data

9. Simplify:
$(x + y)^2 + (y + z)^2 + (z + x)^2 - (x + y + z)^2$

10. Two translations $\mathbf{T}_1$ and $\mathbf{T}_2$ are represented by the vectors $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ respectively. If L is the point (4, 5), find:
(a) the image of L under $\mathbf{T}_1$.
(b) the image of L under $\mathbf{T}_2$.
(c) the image of L under $\mathbf{T}_1$ followed by $\mathbf{T}_2$.

Revision Exercise 2

1. Give the co-ordinates of the images of the following points when rotated through -270^0 about (4, 5):
(a) (–4, 0) (b) (5, 7) (c) (2, –1) (d) (–2, –2)

2. Write down the equations of the following lines, given their gradients and one point on the line:
(a) (0, 0), $-\frac{1}{2}$ (b) (3, 0), 0 (c) (–1, 0), –1 (d) (3, 2), –2

3. The sum of two numbers exceeds a third number by four. If the sum of the three numbers is at least 20 and at most 28, find any three integral values satisfying the inequality.

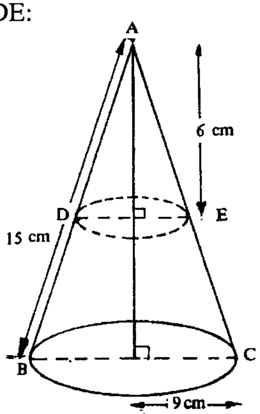
4. Below are records of daily bookings in Karibu Hotel during the month of June 2003:

220	240	190	156	259	260	270	255
199	189	186	220	270	256	256	190
231	238	240	249	256	248	269	
233	245	241	249	257	253	256	

Find the mean number of people booked in the hotel each day.

5. Factorise each of the following expressions:
(a) $4x^2 - 6xy + 4xz + 6yz$ (b) $3p^2 - 8p + 4$
(c) $2s^2 + 3st - 2t^2$ (d) $64k^2 - 49n^2$

6. The volume of a right pyramid with a square base is 256 cm^3. If its height is 16 cm, calculate:
   (a) the base area.
   (b) the side of the square base.
7. Draw the following figures and indicate, where possible, their lines of symmetry:
   (a) Regular hexagon (b) Rhombus
   (c) Square (d) Regular pentagon
   (e) Circle (f) Equilateral triangle
   (g) Trapezium
8. The figure below shows a cone with the vertex at A and diameter BC. The cone is cut off along DE:



   (a) Find the base radius of the smaller cone.
   (b) Find the volume of the frustum.
9. The total surface area of two metallic solid spheres is $116\pi cm^2$ and their radii differ by 3 cm. Find the radius of each sphere. If the two spheres are melted and the material moulded into two spheres of equal radii, find the new total surface area.

Revision Exercise 3

1. The data below is record of time in seconds taken in a 100-metre race by 30 athletes:

10.5	11.4	11.3	12.6	12.8	10.8
10.4	11.1	12.3	12.5	10.9	10.6
11.6	11.5	11.6	11.6	11.9	11.6
10.7	11.0	11.2	12.2	11.5	11.8
13.2	11.2	11.8	12.7	11.0	11.3

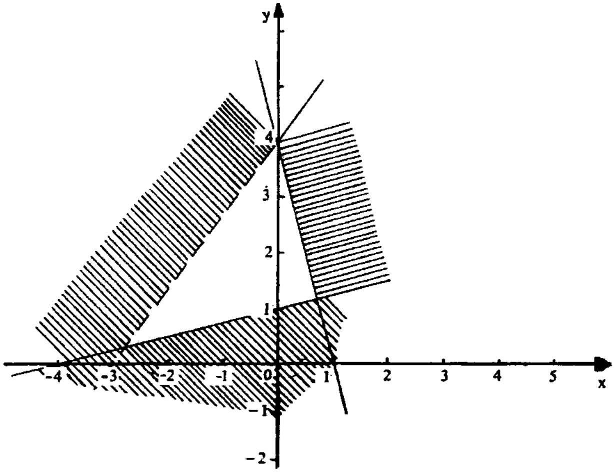
(a) Group the data using class intervals of 0.4.
(b) Find the mean time taken in the race.
(c) Estimate the median of the data.

2. A triangle with vertices at P(2, 3), Q(–3, 3) and R(0, 5) is first rotated about (0, 0) through + 90°, followed by a reflection in the line y = x. If it is finally translated through vector $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$, find the coordinates of the vertices of the image.

3. Find the vertices of a triangle defined by the intersection of the lines:

$y = x$, $\dfrac{y - 1}{x - 5} = -3$ and $3y - 6 = 2 - x$.

4. In the figure below, find the inequalities which describe the unshaded region:



5. The L.C.M. of 15, 18 and a third number is 1 260. Find the square of the third number.

6. A cone has a diameter of 28 cm. height 15 cm and the slant height 18 cm. Calculate:
(a) its total surface area. (b) its volume.

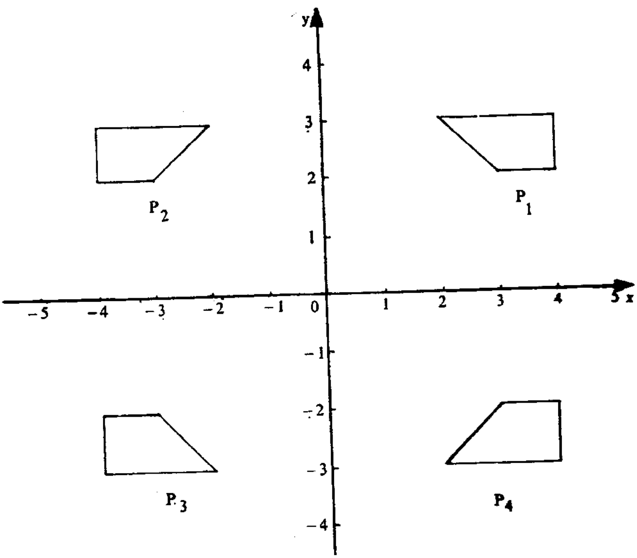
7. Points A(–6, 10), B(–7, 8), C(–6, 6), D(–4, 6), E(–3, 8) and F(–4, 10) are the vertices of a regular hexagon. The hexagon is reflected in the line y = x + 4, followed by a reflection in the line y = – x. Find the co-ordinates of the vertices of the final image of the hexagon.

8. A polygon with vertices A(6, 2), B(6, 4.5), C(5, 4) D(4, 4.5), E(4, 2) and F(5, 2.5) is enlarged by scale factors (i) 2 (ii) – 1 with centre (5, 0). Find the co-ordinates of its image in each case.

9. Solve the equation:
$(p + 1)^2 + 3p - 1 = 0$

Revision Exercise 4

1. Write down the gradients of the following lines (do not draw the lines):
   (a) $3x + 2y - 6 = 0$ (b) $4x = 6 - 2y$ (c) $\frac{x}{3} + \frac{y}{2} = 1$
   (d) $x = -4$ (e) $y = 7$
2. If $2^{x+y} = 16$ and $4^{2x-y} = \frac{1}{4}$, find the ratio of $y - x : 2y$
3. A triangle with the vertices at A(5, – 2), B(3, –4) and C(7,–4) is mapped onto a triangle with the vertices at $A'(5,-12)$, $B'(7,-10)$ and $C'(3,-10)$ by an enlargement. Find the linear scale factor and the centre of the enlargement.
4. Points P(2, 3), Q(4, 5) and R(7, 6) are vertices of triangle PQR. Find the co-ordinates of N, M and D, the midpoints of **PQ**, **QR** and **PR** respectively.
5. Find non-zero scalars λ and μ such that $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{c}$, given that
   $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.
6. Calculate the surface area and volume of a sphere of radius 14 cm.
7. In the figure below, state which of the figures P_1, P_2, P_3 and P_4 are:
   (a) directly congruent.
   (b) indirectly congruent.



8. Solve the simultaneous inequalities: $7x - 8 \leq 342$ and $4x - 1 \geq 135$
9. The masses of 50 babies born in a certain hospital were recorded as below:

Mass (kg)	2.1 – 2.3	2.4 – 2.6	2.7 – 2.9	3.0 – 3.2	3.3 – 3.5	3.6 – 3.8	3.9 – 4.1	4.2–4.4
No. of babies	4	6	11	13	9	2	3	2

(a) Find the mean mass of the babies.
(b) Estimate the median mass.

11. AC is a diameter of a circle ABCD. AB = $(x-2)$ cm, BC = $(x+5)$ cm and the area of ABC is 30 cm^2. Find the radius of the circle. If ACD is isosceles, find its area.

Revision Exercise 5

1. Illustrate the region defined by the following inequalities on graph paper:
   (a) $y \leq 2x + 1$, $y \geq 4 - x$, $xy \geq 2$ and $x \leq 3$.
   (b) $-2 < x \leq 5$, $2y < 3x + 6$, $y \geq x - 5$
2. PQRS is a square. Side QR is extended on either side so that QT = RV. If $\angle PSV = \angle SPT$, show that TS = PV.
3. The internal and external diameters of a spherical shell are 12 cm and 18 cm respectively. Calculate the volume of the material of the shell.
4. Use vectors to show that a triangle with vertices P(2, 3), Q(6, 4) and R(10, 5) is isosceles.
5. The number of vehicles passing through a check-point were recorded for 20 days as follows:

670	840	560	680	674	589
651	496	645	653	605	673
710	702	317	681	627	731
668	726				

Find the mean number of vehicles which passed through this check-point per day.

6. The ratio of the base areas of two cones is 9 : 16.
   (a) Find the ratio of their volumes
   (b) If the larger cone has a volume of 125 cm^3, find the volume of the smaller cone.
7. Use gradient to show that a triangle whose vertices are A(1, 3), B(4, –6) and C(4, 3) is right-angled. Find the area of this triangle.
8. Given that $3^{5x-2y} = 243$ and $3^{2x-y} = 3$, calculate the values of x and y.
9. Find the acute angle between each of the following lines and the x-axis:
   (a) $4y = 12x + 9$ (b) $\frac{2x-3}{4} = \frac{y}{2}$ (c) $0.4x - 0.6y = 5$

10 Factorise $9x^2 - 16y^2 + (3x - 4y)^2$.

Revision Exercise 6

1. The following were the marks scored in English and Mathematics tests by 40 pupils:

Marks	0–9	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89	90–99
English	2	6	6	4	6	11	2	2	1	0
Mathematics	3	3	2	5	7	12	3	2	2	1

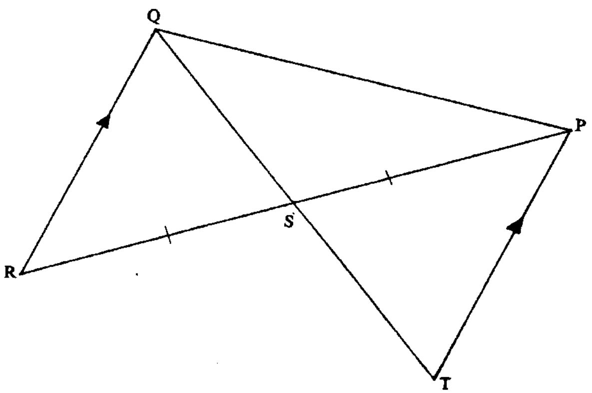
   (a) Draw two histograms on the same axes, showing the marks scored for the two subjects.
   (b) Draw two frequency polygons on the same axes showing the above data.
   (c) Calculate the mean mark in each case.
   (d) What can you say about the scores in Mathematics and English?

2. The depths of two similar buckets are 28 cm and 21 cm respectively. If the larger bucket holds 3.1 litres, find the capacity of the smaller one.

3. Find the equation of a line which is parallel and midway between the lines $\frac{y}{x-3} = 1$ and $\frac{y-2}{x-1} = 1$.

4. Find all the points having integral co-ordinates which satisfy the three in equalities; $y \geq 6 - x$, $y \leq x + 4$ and $x < 4$.

5. In the figure below, RQ//TP and RS = SP. Show that triangles PST and RSQ are congruent.



6. Find the area of a triangular piece of land whose sides are 7 m, 9 m and 14 m.

7. ABCDEFGH is a cube of side 8 cm. If N is the midpoint of AB, find the area of triangle HNG.

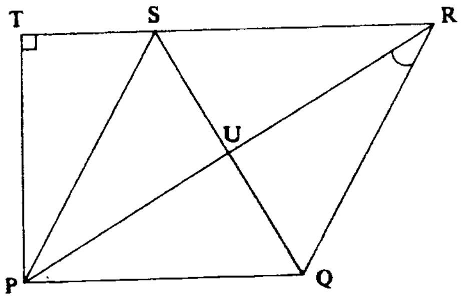
8. Three spacecrafts in different orbits go around the earth at intervals of 3, 6 and 7 hours respectively. An engineer at an observatory on earth first

observes the three crafts cruising above one another at 6.35 a.m. At what time will they be next observed in a similar configuration if the all revolve around the earth from east to west?

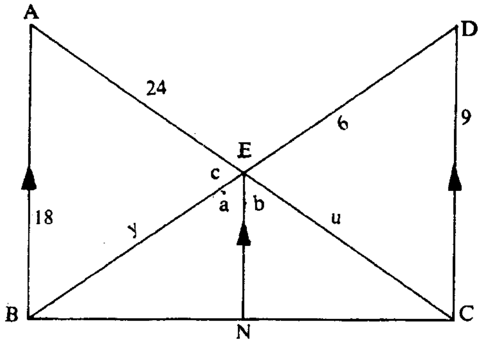
9. Find a if $a^4 - 15a^2 - 34 = 0$.
10. The vertices of a square are P(–3, –1), Q(–1, –1), R(–1, –3) and S(–3, –3). If the square is first reflected in the line y = –x and then translated through a vector $\begin{pmatrix}-1\\4\end{pmatrix}$, find the co-ordinates of the image of the square.

Revision Exercise 7

1. Calculate the radius of a sphere whose volume is 259 cm^3. (Take $\pi = 3.142$)
2. A translation **T** maps a point (4, 1) onto the point (-2, 6). **R** is a half-turn rotation about the point (–1, 2). A triangle with the vertices at P(2, 2), Q(2, –4) and R(–7, –1) is given a translation **T** followed by the rotation **R**. Find the co-ordinates of the first and the final images of the triangle.
3. Draw the region satisfied by the inequalities; $y \geq 0$, $y + x < 5$ and $2x + y \geq 4$.
   (a) State the points of intersection of the boundary lines.
   (b) Find the area of the region.
4. In the figure below, PQRS is a rhombus. If TS = SU, Show that $\angle TPS = \angle PRQ$ and that ΔSUR is congruent to ΔQUP:



5. Find the area of a parallelogram ABCD in which AB = 12 cm, BC = 16 cm and $\angle ABC = 25°$.
6. The mean mass of x objects is x kg and the mean mass of y of them is y kg. Show that the mean mass of the reminder is x + y.
7. In the figure below, name similar triangles and find the lengths of the unknown sides:



8. A triangle whose vertices are A(4, 6), B(3, 4) and C(5, 4) is enlarged with scale factor -1 and centre of enlargement (4, 0). The image is then reflected in the line $y = -x$ followed by an enlargement with linear scale factor -1 and centre (0, 0). Find the co-ordinates of the vertices of the final image.
9. A triangle whose vertices are P(1, 1), Q(2, 1) and R(1.5, 2) is first rotated about (0, 0) through 180^0 followed by an enlargement scale factor 3 with the centre at the origin. Find the co-ordinates of the vertices of the final image.
10. By substituting $x + 2y = p$ in the equation $(x + 2y)^2 + 3(x + 2y) - 4 = 0$, solve the quadratic equation in p. Hence, find possible values of $x + 2y$. If $x - 2y = 4$, find all the possible values of x and y.

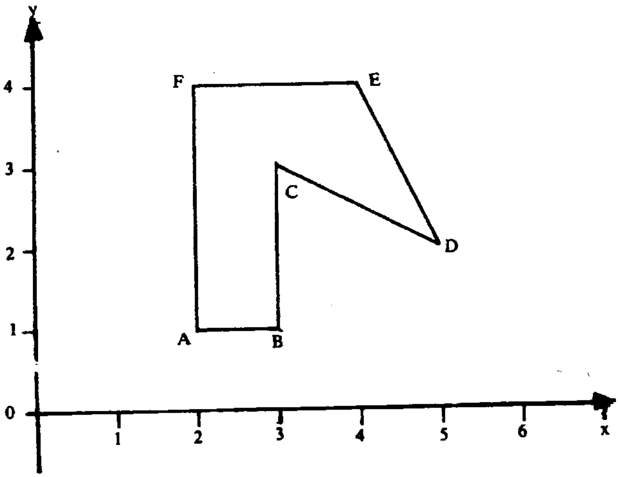
Revision Exercise 8

1. The logarithms of the squares of a and b are 1.204 and 0.954 respectively. Find the logarithm of their product.
2. How many cubes, each of side 19 cm, can be made from a rectangular sheet of metal measuring 2.53 m by 1.39 m?
3. A trapezium ABCD is such that its parallel sides are 18 cm and 26 cm respectively and AB = 9 cm. Find the area of the trapezium if angle ABC = 45^0.
4. Find the area of the region bounded by the inequalities $y \leq \frac{1}{2}(2x + 4)$, $y < 3(2 - x)$ and $y > 0$. If the region is reflected in the line $x = 0$ followed by a reflection in $y = 0$, write down the inequalities satisfying the final image.
5. The length L of a pendulum whose complete swing takes a total time T is given by the formula $L = \frac{T^2 g}{4\pi^2}$. Calculate L when T = 2.31, g = 9.81 and $\pi = 3.142$.

6. Two spheres have surface areas of 36 cm^2 and 49 cm^2 respectively. If the volume of the smaller sphere is 20.2 cm^3, calculate the volume of the larger one.

7. Simplify $\dfrac{(a-c)^2-(a+c)^2}{(a^2+c^2)^2-(a^2-c^2)^2}$. Find the value of the expression when $a = \frac{1}{3}$ and $c = \frac{1}{17}$.

8. The figure below shows an irregular hexagon ABCDEF in the x-y plane:



T is a translation given by vector $\begin{pmatrix}-2\\0\end{pmatrix}$, **Q** is a rotation about (0, 0) through $+90^0$ and **X** is a reflection in the x-axis. Find:

(a) the gradient of the image of AC under **T**.

(b) the equations of CD and ED after **T**, followed by **M**, followed finally by **Q**.

9. A pond holds 27 000 litres of water. How many litres of water would a similar pond hold if its dimensions were double the first one?

10. (a) A shopkeeper makes a profit of Sh. 200 by selling 20 blankets and 50 towels. He makes a loss of Sh. 7 if he sells 35 blankets and 28 towels of the same kind. Determine the profit (or loss) on each of the two items sold.

(b) If he sold 12 blankets, how many towels should he sell to realise an overall profit of sh. 3007?